Appendix D

More Problems

Whatever the details of the matter, it finds me too absorbed by numerous occupations for me to be able to devote my attention to it immediately.

—John Wallis, upon hearing about a problem posed by Fermat in 1657 [Hav03, p. 92]

While realizing that the solution of problems is one of the lowest forms of Mathematical research, and that, in general, it has no scientific value, yet its educational value can not be over estimated. It is the ladder by which the mind ascends into the higher fields of original research and investigation. Many dormant minds have been aroused into activity through the mastery of a single problem.

—Benjamin Finkel and John Colaw on the first page of the first issue of the *American Mathematical Monthly*, 1894

To help readers experience first-hand the excitement, frustration, and joy of working on a challenging numerical problem, we include here a selection in the same style as Trefethen's 10. Of these 22, the two at the end can be considered research problems in the sense that the proposer does not know even a single digit of the answer.

If you solve one of these and wish to share your solution, we will be happy to receive it. We will post, on the web page of this book, solutions that are submitted to us.

1. What is $\sum_n 1/n$, where n is restricted to those positive integers whose decimal representation does not contain the substring 42? (*Folkmar Bornemann*)

2. What is the sum of the series $\sum_{n=1}^{\infty} 1/f(n)$, where $f(1) = 1$, $f(2) = 2$, and if $n > 2$, $f(n) = nf(d)$, with d the number of base-2 digits of n? (*David Smith*)

Remark. Problem A6 of the 2002 Putnam Competition asked for the integers $b \geqslant 2$ such that the series, when generalized to base-b digits, converges.

3. Let $m(k) = k - k/d(k)$ where $d(k)$ is the smallest prime factor of k. What is

$$\lim_{x \uparrow 1} \frac{1}{1-x} \prod_{k=2}^{\infty} \left(1 - \frac{x^{m(k)}}{k+1}\right) \ ?$$

(Arnold Knopfmacher)

Remark. This problem arose from the work of Knopfmacher and Warlimont [KW95]. It is a variation of functions that arise in the study of probabilities related to the irreducible factors in polynomials over Galois fields.

4. If N point charges are distributed on the unit sphere, the potential energy is

$$E = \sum_{j=1}^{N-1} \sum_{k=j+1}^{N} |x_j - x_k|^{-1},$$

where $|x_j - x_k|$ is the Euclidean distance between x_j and x_k. Let E_N denote the minimal value of E over all possible configurations of N charges. What is E_{100}? (*Lloyd N. Trefethen*)

5. Riemann's prime counting function is defined as

$$R(x) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \operatorname{li}(x^{1/k}),$$

where $\mu(k)$ is the Möbius function, which is $(-1)^{\rho}$ when k is a product of ρ different primes and zero otherwise, and $\operatorname{li}(x) = \int_0^x dt/\log t$ is the logarithmic integral, taken as a principal value in Cauchy's sense. What is the largest positive zero of R? (*Jörg Waldvogel*)

Remark. The answer to this problem is truly shocking.

6. Let A be the 48×48 Toeplitz matrix with -1 on the first subdiagonal, $+1$ on the main diagonal and the first three superdiagonals, and 0 elsewhere, and let $\|\cdot\|$ be the matrix 2-norm. What is $\min_p \|p(A)\|$, where p ranges over all monic polynomials of degree 8?

(*Lloyd N. Trefethen*)

7. What is the value of

$$\int_{-1}^{1} \exp\left(x + \sin e^{e^{e^{x+1/3}}}\right) dx \ \ ?$$

(*Lloyd N. Trefethen*)

8. What is the value of

$$\int_0^{\infty} x\, J_0(x\sqrt{2}) J_0(x\sqrt{3}) J_0(x\sqrt{5}) J_0(x\sqrt{7}) J_0(x\sqrt{11})\, dx,$$

where J_0 denotes the Bessel function of the first kind of order zero?

(Folkmar Bornemann)

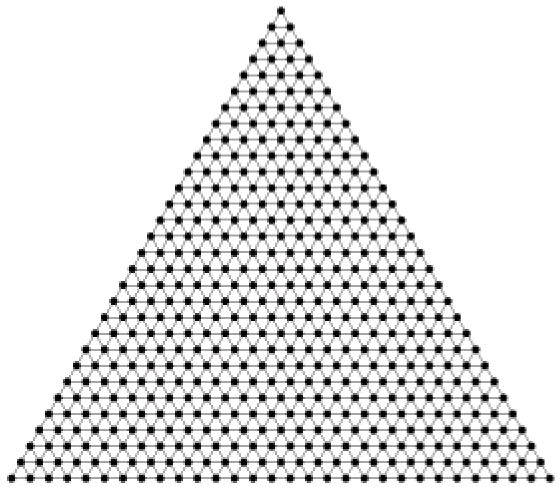

Figure D.1. *A triangular lattice.*

9. If $f(x, y) = e^{-(y+x^3)^2}$ and $g(x, y) = \frac{1}{32}y^2 + e^{\sin y}$, what is the area of the region of the x-y plane in which $f > g$? (*Lloyd N. Trefethen*)

10. The square $c_m(\mathbb{R}^N)^2$ of the least constant in the Sobolev inequality for the domain $\mathbb{R}^N$ is given by the multidimensional integral

$$c_m(\mathbb{R}^N)^2 := (2\pi)^{-N} \int_{\mathbb{R}^N} \left(\sum_{|k| \leqslant m} x^{2k} \right)^{-1} dx,$$

where $k = (k_1, k_2, \ldots, k_N)$ is a multi-index with nonnegative integer elements, and

$$|k| := \sum_{j=1}^{N} k_j, \quad x^k := \prod_{j=1}^{N} x_j{}^{k_j}.$$

For example, we have $c_3(\mathbb{R}^1) = 0.5$. What is $c_{10}(\mathbb{R}^{10})$? (*Jörg Waldvogel*)

11. A particle starts at the top vertex of the array shown in Figure D.1 with 30 points on each side, and then takes 60 random steps. What is the probability that it ends up in the bottom row? (*Lloyd N. Trefethen*)

12. The random sequence x_n satisfies $x_0 = x_1 = 1$ and the recursion $x_{n+1} = 2x_n \pm x_{n-1}$, where each $\pm$ sign is chosen independently with equal probability. To what value does $|x_n|^{1/n}$ converge for $n \to \infty$ almost surely? (*Folkmar Bornemann*)

13. Six masses of mass 1 are fixed at positions $(2, -1)$, $(2, 0)$, $(2, 1)$, $(3, -1)$, $(3, 0)$, and $(3, 1)$. Another particle of mass 1 starts at $(0, 0)$ with velocity 1 in a direction θ (counterclockwise from the x-axis). It then travels according to Newton's laws, feeling an inverse-square force of magnitude r^{-2} from each of the six fixed masses. What is the shortest time in which the moving particle can be made to reach position $(4, 1)$? (*Lloyd N. Trefethen*)

14. Suppose a particle's movement in the x–y plane is governed by the kinetic energy $T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$ and the potential energy

$$U = y + \frac{\epsilon^{-2}}{2}(1 + \alpha x^2)(x^2 + y^2 - 1)^2.$$

The particle starts at the position $(0, 1)$ with the velocity $(1, 1)$. For which parameter α does the particle hit $y = 0$ first at time 10 in the limit $\epsilon \to 0$? (*Folkmar Bornemann*)

15. Let $u = (x, y, z)$ start at $(0, 0, z_0)$ at $t = 0$ with $z_0 \geqslant 0$ and evolve according to the equations

$$\begin{aligned}
\dot{x} &= -x + 10y + \|u\|(-0.7y - 0.03z),\\
\dot{y} &= -y + 10z + \|u\|(0.7x - 0.1z),\\
\dot{z} &= -z + \|u\|(0.03x + 0.1y),
\end{aligned}$$

where $\|u\|^2 = x^2 + y^2 + z^2$. If $\|u(50)\| = 1$, what is z_0? (*Lloyd N. Trefethen*)

16. Consider the Poisson equation $-\Delta u(x) = \exp(\alpha\|x\|^2)$ on a regular pentagon inscribed to the unit circle. On four sides of the pentagon there is the Dirichlet condition $u = 0$, while on one side u satisfies a Neumann condition; that is, the normal derivative of u is zero. For which α does the integral of u along the Neumann side equal e^{α}? (*Folkmar Bornemann*)

17. At what time t_∞ does the solution of the equation $u_t = \Delta u + e^u$ on a 3×3 square with zero boundary and initial data blow up to ∞? (*Lloyd N. Trefethen*)

18. Figure D.2 shows the Daubechies scaling function $\phi_2(x)$ (see [Dau92]) drawn as a curve in the x–y plane. Suppose the heat equation $u_t = u_{xx}$ on the interval $[0, 3]$ is solved with initial data $u(x, 0) = \phi_2(x)$ and boundary conditions $u(0) = u(3) = 0$. At what time does the length of this curve in the x–y plane become 5.4? (*Lloyd N. Trefethen*)

19. Let u be an eigenfunction belonging to the third eigenvalue of the Laplacian with Dirichlet boundary conditions on an L-shaped domain that is made up from three unit squares. What is the length of the zero-level set of u? (*Folkmar Bornemann*)

20. The Koch snowflake is a fractal domain defined as follows. Start with an equilateral triangle with side length 1, and replace the middle third of each side by two sides of an outward-pointing equilateral triangle of side length $1/3$. Now replace the middle third of each of the 12 new sides by two sides of an outward-pointing equilateral triangular of side

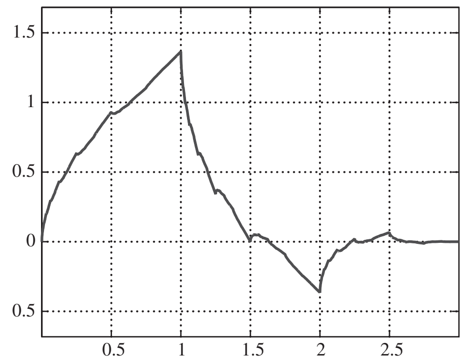

Figure D.2. *Daubechies scaling function* $\phi_2(x)$.

length $1/9$; and so on ad infinitum. What is the smallest eigenvalue of the negative of the Laplacian on the Koch snowflake with zero boundary conditions? (*Lloyd N. Trefethen*)

21. Consider the Poisson equation $-\mathrm{div}(a(x)\mathrm{grad}\, u(x)) = 1$ on the unit square with homogeneous Dirichlet boundary conditions. What is the supremum of the integral of u over the square if $a(x)$ is allowed to vary over all measurable functions that are 1 on half of the area of the square, and 100 on the rest? (*Folkmar Bornemann*)

22. Let $h(z)$ be that solution to the functional equation $\exp(z) = h(h(z))$ which is analytic in a neighborhood of the origin and increasing for real z. What is $h(0)$? What is the radius of convergence of the Maclaurin series of h? (*Dirk Laurie*)

Remark. There are additional properties needed to make h unique. One such simple property has to be found before solving this problem; none is known in the literature right now.