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Exercises 2 and 3 are to be handed in on Thursday, 27.01.2011, before the lecture.

Exercise 1 (Cramer's Rule)

Consider the matrix $A = \begin{pmatrix} 0.2161 & 0.1441 \\ 1.2969 & 0.8648 \end{pmatrix}$ and the vector $b = \begin{pmatrix} 0.1440 \\ 0.8642 \end{pmatrix}$.

(a) Compute $\kappa(A)$ with respect to the $\|.\|_2$-norm (using Matlab).

(b) Explain why $A\tilde{x} - b$ can be seen as a backward error, where $\tilde{x}$ is the numerical solution of $Ax = b$.

(c) Solve the linear system $Ax = b$

  (i) by Cramer's rule, i.e. $x_j = \frac{det(A_j)}{det(A)}$, where A_j is formed by replacing the jth column of A by the column vector b,

  (ii) by using the Matlab function $A \setminus b$.

  Document the forward errors $\|\tilde{x} - x\|_2$ and the backward errors $A\tilde{x} - b$.
  Given this experiment, what is your guess on the stability properties of Cramer's rule for 2×2-matrices?

Exercise 2 (Tridiagonal Systems) (∗)

Consider the linear system $Ax = b$ where $A \in \mathbb{R}^{n\times n}$ is given by a tridiagonal matrix

$$A = \begin{pmatrix} a_1 & c_1 & & 0 \\ e_2 & a_2 & \ddots & \\ & \ddots & \ddots & c_{n-1} \\ 0 & & e_n & a_n \end{pmatrix}.$$

Assume further that the LU factorization of A exists.

(a) Show that the factors L and U must be bidiagonals of the form

$$L = \begin{pmatrix} 1 & 0 & & 0 \\ \beta_2 & 1 & \ddots & \\ & \ddots & \ddots & 0 \\ 0 & & \beta_n & 1 \end{pmatrix}, \quad U = \begin{pmatrix} \alpha_1 & c_1 & & 0 \\ 0 & \alpha_2 & \ddots & \\ & \ddots & \ddots & c_{n-1} \\ 0 & & 0 & \alpha_n \end{pmatrix}$$

with the following recursive relations for the entries of L and U:

$$\alpha_1 = a_1, \; \beta_i = \frac{e_i}{\alpha_{i-1}}, \; \alpha_i = a_i - \beta_i c_{i-1}, \; i = 2, \ldots, n,$$

assuming that the α_i defined in this way are nonzero.
Remark: For a strictly row-dominant matrix A, i.e. $|a_i| > |e_i| + |c_i|$ $\forall i$, this assumption is fulfilled.

(b) What are the solutions of the bidiagonal systems $Ly = b$ and $Ux = y$?

Exercise 3 (LDU Factorization) $(*)$

An LDU factorization of a quadratic matrix $A \in \mathbb{R}^{n\times n}$ is a representation of A as a product $A = LDU$ of three $n \times n$-matrices, where L is lower unit triangular, D is diagonal and U is upper unit triangular

You may use the following fact:

A unique LDU factorization with invertible D of A exists $\Leftrightarrow$ all minors of A are invertible,

where minors are submatrices of the form $A_k = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix} = A(1:k, 1:k),\ k \le n.$

Let A be a symmetric positive definite matrix, i.e. A is a symmetric matrix fulfilling $x^T Ax > 0$ for all $x \in \mathbb{R}^n,\ x \neq 0$.

(a) Show that A has a unique LDU factorization.
Hint: A matrix $M \in \mathbb{R}^{k\times k}$ is invertible $\Leftrightarrow$ there exists no $y \in \mathbb{R}^k,\ y \neq 0$, such that $My = 0$.

(b) Prove that this factorization has the form $A = LDL^T$ (in other words, $U = L^T$).

(c) Prove that the diagonal matrix $D = diag(d_1, \ldots, d_n)$ in the LDU factorization of A is positive definite, i.e. $d_j > 0$ for all $j = 1, \ldots, n$.

(d) Prove that A can be written as $A = \tilde{L}\tilde{L}^T$, where $\tilde{L}$ is lower triangular, $\tilde{L}(i,i) > 0\ \forall i$.