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Winter Semester 2016/17

Numerics of Dynamical Systems

Problem Sheet 5

P5.1 Invariant measures and time-averages

Consider the map

$$f : [0,1] \to [0,1],\ x \mapsto \begin{cases} 2x, & \text{if } x \in [0,1/2] \\ x-1/2, & \text{if } x \in (1/2,1] \end{cases}.$$

(a) Show that f has the invariant density

$$h(x) = \begin{cases} 4/3, & \text{if } x \in [0,1/2] \\ 2/3, & \text{if } x \in (1/2,1] \end{cases}.$$

Hint: It suffices to consider sets which are intervals.

(b) Show directly, using binary representation and the law of large numbers, that for a random number x chosen from the uniform distribution on $[0,1]$

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=0}^{N-1} \chi_{[0,1/2]} \left(f^j(x)\right) = \frac{2}{3}$$

with probability 1. Do not use Birkhoff's Ergodic Theorem.

P5.2 Invariant density of the logistic map

The logistic map is defined as

$$S : [0,1] \to [0,1], \quad S(x) \ = \ \mu x(1-x), \quad \mu \in [3,4].$$

(a) Plot trajectories of length 1000 for different values $\mu \in [3,4]$ and initial points $x_0 \in [0,1]$.

(b) To visualize the dependence of the trajectories on the parameters, write a MATLAB script `bifurc.m` which generates trajectories of length 1000 for varying values of μ, with a randomly generated initial value x_0 for each μ. Here, μ shall increase, starting at $\mu = 3$, with increment $\Delta\mu = 0.02$ to the maximal value $\mu = 4$.

Finally, create a μ-x-plot, i.e. the script shall plot the individual points of the trajectory vertically over the corresponding value of μ.

(c) Now, we want to consider the system with parameter $\mu = 4$ in more detail.

Again, generate a trajectory and plot the histogram (MATLAB function `hist`) over 100 "bins".

Try different numbers of iterations and compare the result to the exact invariant density

$$h^*(x) \ = \ \frac{1}{\pi\sqrt{x(1-x)}}.$$