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NUMERICS OF DYNAMICAL SYSTEMS

Problem Sheet 3

P3.1 Application of the subdivision algorithm: Zero-finding

Let $f : \mathbb{R}^d \to \mathbb{R}^d$ be continuously differentiable. Zeros $\bar{x}$ of f, i.e. $f(\bar{x}) = 0 \in \mathbb{R}^d$, can be found using the (iterative) Newton method: For an initial value x_0 the sequence

$$x_{n+1} = x_n - \left(J_f(x_n)\right)^{-1} f(x_n) \tag{1}$$

converges quadratically to a zero of f, where $J_f(x_n) = \frac{\partial}{\partial x} f(x)\Big|_{x=x_n}$ is the Jacobian of f in the point x_n.

The iteration (1) defines a discrete dynamical system. Its fixed points are the zeros of f. The subdivision algorithm, applied on (1), constructs a covering of the zeros in a certain region.
Consider now the function $f : \mathbb{R}^2 \to \mathbb{R}^2$,

$$f(x,y) = \begin{pmatrix} 4x(x^2+y-11) + 2(x+y^2-7) \\ 2(x^2+y-11) + 4y(x+y^2-7) \end{pmatrix}.$$

(a) Modify the algorithm from P2.2 such that it calculates a covering of the zeros of f in a rectangle $Q \subset \mathbb{R}^2$.
(b) Use the subdivision algorithm to approximate the zeros in the square $[-5,5]^2$.
How would you change the algorithm to converge to the zeros faster?

P3.2 The Ikeda system in GAIO

We consider the two-dimensional Ikeda map

$$f(x,y) = \begin{pmatrix} a + b(x\cos(\theta) + y\sin(\theta)) \\ b(x\sin(\theta) - y\cos(\theta)) \end{pmatrix}$$

with $\theta = c - \frac{d}{x^2+y^2+1}$ and the parameters $a = 2.16,\ b = 0.65,\ c = 4.96,\ d = 37.65$.

(a) Show that the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism.
(b) Plot the first 10000 points of the (discrete) trajectory $f^j(1,1),\ j = 1,\dots,10000$ for the initial point $(1,1)$. Compare the behaviour for different initial points.
(c) Download the file `attractor.m` from the course website[1] and modify it so that it approximates the relative global attractor of the Ikeda map w.r.t. the set[2] $Q = [-4,6]^2$ up to depth 12, with 16 sampling points per box (i) on the edges of the boxes, (ii) in a grid, or (iii) chosen randomly. (Use `help Tree` to get a description of the GAIO methods.) Compare the approximated attractor with the plot from (b).

P3.3 Convergence speed of the subdivision algorithm

For the linear contraction $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \mu x$, $0 < \mu < 1$, consider the computation of the relative global attractor of $[-1,1]$ using the subdivision algorithm (with division of intervals at their midpoints).

(a) Compute the number α in the convergence theorem 2.18.
(b) How many iterations of f are necessary to ensure $\alpha < 1$ if (i) $\mu = 1/4$, (ii) $\mu = 3/4$?
(c) What happens if $\alpha \geq 1$?

[1] http://www-m3.ma.tum.de/NumDynSys1617/
[2] Please note that GAIO uses center and radius to describe a box.