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Numerics of Dynamical Systems

Problem Sheet 2

P2.1 Global attractor

We consider the map

$$f : \mathbb{R} \to \mathbb{R}, \quad f(x) = x^3$$

(a) Find the global attractor A of the discrete dynamical system $x_{n+1} = f(x_n)$.

(b) Find A_Q, the global attractor relative to $Q = [-\frac{1}{2}, \frac{3}{2}]$. Compare it to A.

P2.2 Implementation of the subdivision algorithm

In this exercise, we will implement the subdivision algorithm from the lecture for the two-dimensional *Hénon map* in MATLAB. Some auxiliary functions (see page 2) necessary for this exercise can be found on the course website[1].

The Hénon map is defined by

$$f : \mathbb{R}^2 \to \mathbb{R}^2, \quad f : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \mapsto \begin{pmatrix} x_2 + 1 - a x_1^2 \\ b x_1 \end{pmatrix}, \tag{1}$$

where we use the values 1.4 resp. 0.3 for the parameters a and b.

The objects appearing in the algorithm should be stored as follows:

- We approximate the global attractor of the system relative to a rectangular area $Q \subset \mathbb{R}^2$. Q is defined by its bottom left and top right corners $x_{\min}$ resp. $x_{\max}$, stored as column vectors.
- The subdivision will proceed by dividing each parent rectangle into four equally sized child rectangles. Hence, a subdivision of Q at the kth level is a $2^k \times 2^k$-grid of boxes.
- The collection $\mathcal{B}_k$ will be stored as a $2^k \times 2^k$-matrix T with $T_{i,j} = 1$ if the (i, j)th box at level k is a part of $\mathcal{B}_k$ and $T_{i,j} = 0$ if it is not.

Proceed as follows for the implementation:

(a) Write a function `samplePoint(x_min, x_max, depth, box)` which chooses a random point from the specified subrectangle of Q at level `depth` (uniformly distributed).

(b) Write a function `rgaHenon(x_min, x_max, maxdepth, s)`, which implements the subdivision algorithm up to depth `maxdepth`. To (approximately) determine whether $f(B') \cap B \neq \emptyset$, draw a set S of s random points from B' and check whether $f(S) \cap B \neq \emptyset$. Try to find an efficient implementation for this check.

(c) Approximate the relative global attractor of the set $Q = [-2, 2]^2$ up to depth 7 with 100 sample points per box. Visualize it with the function `drawBoxes`.

[1] http://www-m3.ma.tum.de/NumDynSys1617/

You can use the following auxiliary functions:

henon: the Hénon map.

subdivide: computes $\widehat{\mathcal{B}_k}$ by subdividing the elements of a collection $\mathcal{B}_k$.

coordToBox: calculates from the cartesian coordinates of a point the indices of the box in which it lies.

boxToCoord: calculates from the indices of a box the cartesian coordinates of the corners $x_{\min}$ and $x_{\max}$.

drawBoxes: draws the elements of a collection $\mathcal{B}_k$.

P2.3* [Optional:] Global unstable manifolds

From the lecture, we have the following algorithm for computing one-dimensional unstable manifolds:

$R := [x_0,\ x_1 = g(x_0)]$
$x_2 := g^2(x_0)$
$M := \{\bar{x}\} \cup R$
for $k = 1, 2, \ldots$ **do**
  if $\|x_1 - x_2\| < TOL$ **then**
    $M := M \cup \{x_2\}$
    $R := [R(2:\mathrm{end}),\ x_2]$
    $x_1 := x_2$
    $x_2 := g(R(2))$
  else
    $R := [R(1),\ (R(1)+R(2))/2,\ R(2:\mathrm{end})]$
    $x_2 := g(R(2))$
  end
end

In this exercise, we will show that the algorithm can not get stuck in the interpolation step. We assume g to be continuous.

(a) Let $R(2) = r$ after some step k of the algorithm. Show that the algorithm will eventually accept with $x_2 = g(r)$.

(b) Conclude that $g^n(x_0) \in M$ after a finite number of steps for all $n \in \mathbb{N}$.