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NUMERICS OF DYNAMICAL SYSTEMS

Problem Sheet 1

P1.1 Phase portraits

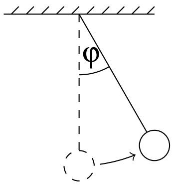
Find and classify the equilibrium points of the following dynamical systems. Use your results to sketch the phase portraits and verify your solution with the MATLAB-function `quiver`.

(a) $\dot{x} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} x$

(b) $\dot{x} = -x + x^2$
$\dot{y} = x + y$

P1.2 Mathematical pendulum

The ordinary differential equation



$$\ddot{\varphi} + \sin\varphi = 0 \quad \Longleftrightarrow \quad \begin{cases} \dot{\varphi} = v \\ \dot{v} = -\sin\varphi \end{cases}$$

describes a frictionless plane pendulum (where φ is the angle relative to the position at rest).

(a) Find the equilibrium points and their stable, unstable, resp. center eigenspaces. Can the linearization be used to determine the stability of the equilibrium $(0,0)$?

(b) Prove the following theorem:

Let $\bar{x}$ be a fixed point of $\dot{x} = f(x)$ and U a neighborhood of $\bar{x}$. If there is a C^1-function $V : U \to \mathbb{R}$ such that

i) $V(\bar{x}) = 0,\ V(x) > 0$ for all $x \in U \backslash \{\bar{x}\}$ and

ii) the *Lie-derivative* $\dot{V}(x) := \nabla V(x) \cdot f(x)$ fulfills $\dot{V}(x) \leq 0$ for all $x \in U \backslash \{\bar{x}\}$,

then $\bar{x}$ is a stable fixed point. (In this case, V is called a *Lyapunov function* of the system at $\bar{x}$.)

(c) Show that $V(\varphi, v) = \frac{1}{2}v^2 - \cos\varphi + 1$ is a Lyapunov function of the system at $(0,0)$ and consequently that the origin is a stable fixed point.

Which physical quantity does V correspond to?

P1.3 Non-wandering points

Let $x \mapsto f(x)$ be a discrete dynamical system. A point x is called *non-wandering* if, for every neighborhood U of x and every $N \in \mathbb{N}$, there is an $n \geq N$ such that $f^n(U) \cap U \neq \emptyset$. Let Ω be the set of all non-wandering points of f. Show that:

- Ω is closed.
- Ω is invariant (i.e. $f(\Omega) = \Omega$) if $f : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism.
- $\overline{\mathrm{Per}(f)} \subset \Omega$, where $\mathrm{Per}(f)$ is the set of periodic points of f.
- Not every invariant set consists of non-wandering points. Consider the system $x \mapsto \begin{pmatrix} 1 & 1 \\ 0 & \frac{1}{2} \end{pmatrix} x$ to find a counterexample.