11 H21 b) Finde alle $f \in H(\mathbb{C})$ mit

$(z = x+iy)$ $\qquad \mathrm{Re}(f) = ax^2 + by^2 \qquad a, b \in \mathbb{R}$

Lösung

$$\frac{\partial v}{\partial x} \overset{CR}{=} -\frac{\partial u}{\partial y} = -(2by)$$

$$\Rightarrow v(x,y) = \underbrace{v(0,y)}_{=:C(y)} + \int_0^x \frac{\partial v}{\partial x}(s,y)\,ds$$

$$= C(y) + \int_0^x (-2by)\,ds = C(y) - 2bxy$$

$$\left.\begin{array}{l} \Rightarrow \dfrac{\partial v}{\partial y} = C'(y) - 2bx \\ \quad \| CR \\ +\dfrac{\partial u}{\partial x} = 2ax \end{array}\right\} \Rightarrow 2ax = C'(y) - 2bx$$

$$2(a+b)x = \underbrace{C'(y)}_{\text{unabhängig von } x}$$

$$a+b=0 \Leftarrow$$

$$\Rightarrow C'(y) = 0 \Rightarrow C(y) \text{ konstant}$$

$$\Rightarrow v(x,y) = C(y) - 2bxy = \overset{\mathbb{R}\ni}{C} - 2bxy$$

$$\Rightarrow f(z) = u + iv = ax^2 + by^2 + i(C - 2bxy)$$

$$\overset{b=-a}{=} ax^2 - ay^2 + iC + 2iaxy$$

$$= a(x^2 + 2xiy - y^2) + iC$$

$$= a(x^2 + 2x(iy) + (iy)^2) + iC$$

$$= a(x+iy)^2 + iC$$

$$= \underline{\underline{az^2 + iC}} \qquad a, C \in \mathbb{R} \text{ beliebig}$$