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The singular values of the GOE

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As a unifying framework for examining several properties that nominally involve eigenvalues, we present a particular structure of the singular values of the Gaussian orthogonal ensemble (GOE): the even-location singular values are distributed as the positive eigenvalues of a Gaussian ensemble with chiral unitary symmetry, while the odd-location singular values, conditioned on the even-location ones, can be algebraically transformed into a set of independent χ-distributed random variables. We discuss three applications of this structure: first, there is a pair of bidiagonal square matrices, whose singular values are jointly distributed as the even- and odd-location ones of the GOE; second, the magnitude of the determinant of the GOE is distributed as a product of simple independent random variables; third, on symmetric intervals, the gap probabilities of the GOE can be expressed in terms of the Laguerre unitary ensemble. We work specifically with matrices of finite order, but by passing to a large matrix limit, we also obtain new insight into asymptotic properties such as the central limit theorem of the determinant or the gap probabilities in the bulk-scaling limit. The analysis in this paper avoids much of the technical machinery (e.g. Pfaffians, skew-orthogonal polynomials, martingales, Meijer G-function, etc.) that was previously used to analyze some of the applications.

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1. Introduction

This paper studies the structure of the singular values of the Gaussian orthogonal ensemble (GOE), using it as a unifying framework for examining several properties that nominally involve eigenvalues. Here, the GOE_n of order n is the ensemble of real symmetric random matrices

$$G = (X + X')/2,$$

where X is an $n \times n$ Gaussian matrix with all entries independent standard normals. Since the singular values of symmetric matrices are the magnitudes of the eigenvalues, the ensemble of singular values will be briefly denoted by $|\mathrm{GOE}_n|$. Central to our discussion is the immediate set decomposition,

$$|\mathrm{GOE}_n| = \mathrm{even}|\mathrm{GOE}_n| \cup \mathrm{odd}|\mathrm{GOE}_n|, \tag{1.1}$$

of the ordered singular values according to the parity of their indices, where the even-location decimated ensemble $\mathrm{even}|\mathrm{GOE}_n|$ is defined by taking the second largest, fourth largest, etc. singular value, and similarly for $\mathrm{odd}|\mathrm{GOE}_n|$.

Our *first* set of main results relates the decomposition (1.1) to the eigenvalues of a Gaussian ensemble with chiral, or anti-symmetric, unitary symmetry. Namely, with X as above, the ensemble of real skew-symmetric random matrices

$$A = (X - X')/2$$

will be called the anti-GUE with its (almost surely) different and positive singular values written briefly as aGUE_n (if n is odd, there is a surplus singular value zero, which is omitted).[a] Then, the following structure holds.

Theorem 1.1. *Denoting equality of the joint distribution by* $\stackrel{\mathrm{d}}{=}$, *there holds*

$$\mathrm{even}|\mathrm{GOE}_n| \stackrel{\mathrm{d}}{=} \mathrm{aGUE}_n. \tag{1.2}$$

We will give two proofs that differ in their handling of the odd-location singular values: one (Sec. 4) by algebraically transforming them to a set of *independent* positive variables, each distributed as χ_2 and, if n is odd, a surplus χ_1; the other (Sec. 7) by integrating them out. Both proofs are based on an algebraic factorization (Sec. 2) of the joint density of $|\mathrm{GOE}_n|$, where one factor depends only on the even-location singular values, the other on the odd-location ones. If we recall the superposition representation, see [8, Eq. (2.6)] or [7, Theorem 1],

$$|\mathrm{GUE}_n| \stackrel{\mathrm{d}}{=} \mathrm{aGUE}_n \cup \mathrm{aGUE}_{n+1},$$

of the singular values of the Gaussian unitary ensemble (GUE), with both ensembles on the right drawn independently, Theorem 1.1 immediately implies the following remarkable relation between the singular values of GUE and GOE.

Corollary 1.1. *With the ensembles on the right drawn independently, there holds*

$$|\mathrm{GUE}_n| \stackrel{\mathrm{d}}{=} \mathrm{even}|\mathrm{GOE}_n| \cup \mathrm{even}|\mathrm{GOE}_{n+1}|. \tag{1.3}$$

[a] In this paper, the Gaussian weights are $e^{-\beta x^2/2}$ with $\beta = 1$ for orthogonal and $\beta = 2$ for unitary symmetry.

The superposition (1.3) bears a striking similarity with a corresponding result for the eigenvalue distributions, see [10, Eq. (5.9); 11, Eq. (6.14)], namely

$$\mathrm{GUE}_n \overset{\mathrm{d}}{=} \mathrm{even}(\mathrm{GOE}_n \cup \mathrm{GOE}_{n+1}).$$

Our *second* set of main results (Secs. 4 and 5) sharpens Theorem 1.1 by realizing $|\mathrm{GOE}_n|$ as the singular values of other matrix models that reveal a rich additional structure. A first model (Corollary 4.1), initially identified by comparing moments of the product of the even singular values to known moments of the determinant [1, Eqs. (23) and (24)], is constructed by bordering the skew-symmetric matrix A defining the anti-GUE with an independent standard normal vector $b \in \mathbb{R}^n$: that is to say, the singular values of G and those of $H = (b\ A)$ are both distributed as $|\mathrm{GOE}_n|$. To the same end, the bordering vector could also be chosen as $b = \tau_n e_1$, where τ_n is a χ_n-distributed variable, independent of A, and e_1 denotes the first unit vector. The precise effect of such borderings on the singular values of a matrix is studied in Sec. 3.

Using a technique (Lemma 5.1) that was, in essence, introduced by Dumitriu and Forrester [6], this bordered matrix model is finally (Theorems 5.1 and 5.2) transformed into a pair $(R^{\mathrm{even}}, R^{\mathrm{odd}})$ of *bidiagonal* square matrices, whose singular values are jointly distributed as $\mathrm{even}|\mathrm{GOE}_n|$ and $\mathrm{odd}|\mathrm{GOE}_n|$. Both matrices depend in a very simple fashion on a set of independent χ_k-distributed random variables. Specifically, for $n = 2m$ even (the structure of the odd order case is similar), we get

$$R^{\mathrm{odd}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\xi_1^2 + 2\xi_{2m}^2} & \xi_{2m-2} & & & \\ & \xi_{2m-1} & \xi_{2m-4} & & \\ & & \ddots & \ddots & \\ & & & \xi_5 & \xi_2 \\ & & & & \xi_3 \end{pmatrix}$$

and

$$R^{\mathrm{even}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_1 & \xi_{2m-2} & & & \\ & \xi_{2m-1} & \xi_{2m-4} & & \\ & & \ddots & \ddots & \\ & & & \xi_5 & \xi_2 \\ & & & & \xi_3 \end{pmatrix},$$

where $\xi_1, \xi_2, \ldots, \xi_{2m}$ are independent random variables, with ξ_k distributed as χ_k. The singular values of R^{odd} correspond to $\mathrm{odd}|\mathrm{GOE}_{2m}|$ and those of R^{even} to $\mathrm{even}|\mathrm{GOE}_{2m}|$, both drawn from the same ensemble.

As a striking application (Corollary 5.1) of this new matrix pair model, we establish that $|\det \mathrm{GOE}_n|$ can be expressed explicitly as a product of independent random variables. Specifically, for $n = 2m$ even, the determinant of $M = \sqrt{2}G$

factors as

$$|\det M| = \xi_1 \sqrt{\xi_1^2 + 2\xi_{2m}^2} \cdot \xi_3^2 \cdot \xi_5^2 \cdots \xi_{2m-1}^2, \tag{1.4}$$

with independent variables ξ_k distributed as χ_k. A similar factorization holds in the odd order case. The form of these variables explains the absence of large prime factors in the moments of the determinant, and leads to a new, simple proof (Sec. 6) of the known central limit theorem for $\log|\det \mathrm{GOE}_n|$, cf. [4, Sec. III; 17, Theorem 4]. While the representation (1.4) of $|\det M|$ as a product of independent random variables can be found implicitly in the work of Delannay and Le Caër, namely in form of a factorization [4, Eq. (41)] of the Meijer G-function representation of the Mellin transform of $|\det M|$ into hypergeometric terms, see the discussion of (5.6), Tao and Vu, who approximated the log-determinant by a sum of weakly dependent terms, speculated that such a representation would not be possible [17, p. 78].

Our *third* set of main results (Sec. 8) studies the implication of Theorem 1.1 on the inter-relation of gap probabilities, that is, the probabilities $E(k; J)$ that the interval J contains exactly k eigenvalues drawn from a random matrix ensemble. Specifically, for order n, we get

$$E^n_{\mathrm{GOE}}(2k + \mu - 1; (-s, s)) + E^n_{\mathrm{GOE}}(2k + \mu; (-s, s)) = E^n_{\mathrm{aGUE}}(k; (0, s)),$$

where $\mu = 0, 1$ denotes the parity of n. This formula was previously known only in the case $\mu = 0$, see [8, Eq. (1.14)]. We initially used a heuristic argument, see (8.3), to extrapolate the formula to the case $\mu = 1$. A substantial portion of the present discussion was derived from attempts to justify the heuristic, after this prediction held up under numerical scrutiny. Taking the bulk scaling limit of both cases provides a new, simpler proof of a remarkable formula previously obtained by Mehta relating the gap probabilities of the GOE and those of the Laguerre unitary ensemble (LUE), see (8.4).

Notation. In contrast to the previous analyses mentioned, where either ensembles of odd (e.g. if Pfaffians were used) or of even order (e.g. if Mellin transforms were used) have typically presented considerable technical complications, our treatment of ensembles of even and odd order is nearly identical. The formulae themselves, however, will often depend on the parity μ of the underlying order n and we will, throughout this paper, write

$$n = 2m + \mu \quad (\mu = 0, 1), \quad \hat{m} = m + \mu, \tag{1.5a}$$

that is,

$$m = \lfloor n/2 \rfloor, \quad \hat{m} = \lceil n/2 \rceil, \quad \mu = \lceil n/2 \rceil - \lfloor n/2 \rfloor. \tag{1.5b}$$

Terms that only appear for n odd will be written with a factor μ in a sum and with an exponent μ in a product; etc. This way, without suggesting any natural interpolation between the cases $\mu = 0$ and $\mu = 1$ (with the notable exception of the usage of the heuristic duality principle (8.3) that started our work), we simply avoid writing out awkward case distinctions.

2. Joint Density of the Singular Values

In this section we establish, in two different ways, the joint probability distribution of the singular values $\sigma_j = |\lambda_j|$ of the GOE induced by the corresponding density for eigenvalues as given by the symmetric function

$$p(\lambda_1, \lambda_2, \ldots, \lambda_n) = c_n \prod_{k=1}^{n} \mathrm{e}^{-\lambda_k^2/2} \cdot |\Delta(\lambda_1, \lambda_2, \ldots, \lambda_n)| \tag{2.1}$$

with some normalization constant c_n and the Vandermonde determinant

$$\Delta(\xi_1, \ldots, \xi_n) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \xi_1 & \xi_2 & \cdots & \xi_n \\ \vdots & \vdots & & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \cdots & \xi_n^{n-1} \end{pmatrix} = \prod_{k>j} (\xi_k - \xi_j).$$

We will frequently use that $\Delta(\xi_1, \ldots, \xi_n) \geq 0$ if the arguments are increasingly ordered, $\xi_1 \leq \cdots \leq \xi_n$.

By symmetry, we can establish the joint density of the singular values by restricting ourselves to the cone of increasingly ordered singular values

$$0 \leq \sigma_1 \leq \cdots \leq \sigma_n, \tag{2.2}$$

this way parametrizing $|\mathrm{GOE}_n|$. To simplify notation and avoid case distinctions between odd and even order n in later parts of the paper, we introduce two further sets of coordinates for this cone. Writing, as detailed in (1.5), $n = 2m + \mu$ and $\hat{m} = m + \mu$ with $\mu = 0, 1$, the coordinates

$$x_j = \sigma_{2j-1} \quad (j = 1, \ldots, \hat{m}), \quad y_j = \sigma_{2j} \quad (j = 1, \ldots, m) \tag{2.3a}$$

satisfy the interlacing property

$$0 \leq x_1 \leq y_1 \leq x_2 \leq y_2 \leq \cdots \leq x_{\hat{m}} \leq y_{\hat{m}}, \tag{2.3b}$$

formally adding the value $y_{m+1} = \infty$ if $\mu = 1$. With $x^{\downarrow}$ and $y^{\downarrow}$ denoting the x and y vectors with their components taken in the reverse order, so $x^{\downarrow} = (x_{\hat{m}}, x_{\hat{m}-1}, \ldots, x_1)$ and $y^{\downarrow} = (y_m, y_{m-1}, \ldots, y_1)$, we define, depending on the parity of n, the coordinates

$$(t, s) = (y^{\downarrow}, x^{\downarrow}) \quad (\mu = 0), \quad (t, s) = (x^{\downarrow}, y^{\downarrow}) \quad (\mu = 1), \tag{2.4a}$$

satisfying the interlacing property

$$t_1 \geq s_1 \geq t_2 \geq s_2 \geq \cdots \geq t_{\hat{m}} \geq s_{\hat{m}} \geq 0, \tag{2.4b}$$

again formally adding the value $s_{m+1} = 0$ if $\mu = 1$. A large part of the apparent dependence on parity is the fact that some results, like Theorem 2.1, have stable expressions in terms of the (x, y) coordinates, while others, like Theorem 4.1, are stable in the (t, s) coordinates. Since the mapping from $\sigma = (\sigma_1, \ldots, \sigma_n)$ to either the pair of coordinates (x, y) or (t, s) is orthogonal, transforming the density between the three sets of coordinates is simply done by inserting new variable names for old ones. Note that the s variables parametrize the even-location decimated ensemble even$|\mathrm{GOE}_n|$ while the t-variables do the same for odd$|\mathrm{GOE}_n|$. We call them the even and odd singular values.

Supported on the cone defined by (2.2), the joint probability density of the singular values is

$$q(\sigma_1,\ldots,\sigma_n) = n! \sum_{\epsilon\in\{\pm 1\}^n} p(\epsilon_1\sigma_1,\ldots,\epsilon_n\sigma_n) = c_n n! \prod_{k=1}^{n} e^{-\sigma_k^2/2} D(\sigma_1,\ldots,\sigma_n)$$

with

$$D(\sigma_1,\ldots,\sigma_n) = \sum_{\epsilon\in\{\pm 1\}^n} |\Delta(\epsilon_1\sigma_1,\ldots,\epsilon_n\sigma_n)|.$$

To determine the signs of the Vandermonde terms it suffices to discuss the case $\sigma_1 < \cdots < \sigma_n$: we then get, because $\operatorname{sign}(\epsilon_k\sigma_k - \epsilon_j\sigma_j) = \epsilon_k$ if $k > j$,

$$\operatorname{sign}\Delta(\epsilon_1\sigma_1,\ldots,\epsilon_n\sigma_n) = \prod_{k>j}\epsilon_k = \prod_{k=2}^{n}\epsilon_k^{k-1} = \prod_{k\text{ even}}\epsilon_k.$$

Hence, by continuity, there holds on all of (2.2)

$$D(\sigma_1,\ldots,\sigma_n) = \sum_{\epsilon\in\{\pm 1\}^n} \theta_0(\epsilon)\,\Delta(\epsilon_1\sigma_1,\ldots,\epsilon_n\sigma_n), \quad \theta_0(\epsilon) = \prod_{k\text{ even}}\epsilon_k. \tag{2.5}$$

The form of $\theta_0(\epsilon)$ suggests we proceed in terms of the (x, y) coordinates introduced in (2.3). With respect to these coordinates, we obtain the following theorem.

Theorem 2.1. *The joint probability density of* $|\mathrm{GOE}_n|$, *supported on the cone* (2.2) *and expressed in the coordinates* (2.3), *is given by*

$$c_n n! 2^n \cdot \left(\prod_{k=1}^{\hat m} e^{-x_k^2/2}\cdot\Delta(x_1^2,\ldots,x_{\hat m}^2)\right)\cdot\left(\prod_{k=1}^{m} y_k e^{-y_k^2/2}\cdot\Delta(y_1^2,\ldots,y_m^2)\right), \tag{2.6}$$

where c_n is the normalization constant of the GOE-*density* (2.1).

Remark 2.1. Despite the fact that the joint density factors on its domain of support, it does *not* reveal an independence between the underlying variables x and y. Their dependence is entirely by the interlacing (2.3b).

We will give two different proofs of the theorem. The first uses the determinantal structure of the Vandermonde terms to establish the factorization, while the second uses their polynomial structure and their symmetries. It is our consideration that the first proof is more straightforward, while the second provides additional insight into the structure of the factorization.

Proof by Determinantal Structure

We write $D(x;y)$ for (2.5) when expressed in terms of the (x,y) variables (2.3); it is convenient to split the sign changes ϵ into ϵ^x and ϵ^y accordingly and use

$$\theta_0(\epsilon) = \theta(\epsilon^y), \quad \theta(\epsilon^y) = \epsilon_1^y \cdots \epsilon_m^y.$$

Since determinants are invariant with respect to a simultaneous row and column permutation so that the odd columns and rows occur before the even ones, we express the Vandermonde terms as

$$\Delta(\epsilon\sigma_1, \ldots, \epsilon\sigma_n) = \det\begin{pmatrix} \pi_0^{(\hat{m})}(x_1) & \ldots & \pi_0^{(\hat{m})}(x_{\hat{m}}) & \pi_0^{(\hat{m})}(y_1) & \ldots & \pi_0^{(\hat{m})}(y_m) \\ \epsilon_1^x\pi_1^{(m)}(x_1) & \ldots & \epsilon_{\hat{m}}^x\pi_1^{(m)}(x_{\hat{m}}) & \epsilon_1^y\pi_1^{(m)}(y_1) & \ldots & \epsilon_m^y\pi_1^{(m)}(y_m) \end{pmatrix}$$

by writing the determinant column-wise with

$$\pi_\mu^{(n)}(x) = \begin{pmatrix} x^\mu \\ x^{\mu+2} \\ \vdots \\ x^{\mu+2n-2} \end{pmatrix} \in \mathbb{R}^n \quad (\mu = 0, 1).$$

Now, we calculate

$$\begin{aligned} D(x;y) &= \sum_{\substack{\epsilon^x \in \{\pm 1\}^{\hat{m}} \\ \epsilon^y \in \{\pm 1\}^m}} \theta(\epsilon^y) \det\begin{pmatrix} \pi_0^{(\hat{m})}(x_1) & \ldots & \pi_0^{(\hat{m})}(x_{\hat{m}}) & \pi_0^{(\hat{m})}(y_1) & \ldots & \pi_0^{(\hat{m})}(y_m) \\ \epsilon_1^x\pi_1^{(m)}(x_1) & \ldots & \epsilon_{\hat{m}}^x\pi_1^{(m)}(x_{\hat{m}}) & \epsilon_1^y\pi_1^{(m)}(y_1) & \ldots & \epsilon_m^y\pi_1^{(m)}(y_m) \end{pmatrix} \\ &= \sum_{\epsilon^y \in \{\pm 1\}^m} \theta(\epsilon^y) \det\begin{pmatrix} 2\pi_0^{(\hat{m})}(x_1) & \ldots & 2\pi_0^{(\hat{m})}(x_{\hat{m}}) & \pi_0^{(\hat{m})}(y_1) & \ldots & \pi_0^{(\hat{m})}(y_m) \\ 0 & \ldots & 0 & \epsilon_1^y\pi_1^{(m)}(y_1) & \ldots & \epsilon_m^y\pi_1^{(m)}(y_m) \end{pmatrix} \\ &= 2^{\hat{m}} \det\big(\pi_0^{(\hat{m})}(x_1) \quad \ldots \quad \pi_0^{(\hat{m})}(x_{\hat{m}})\big) \\ &\quad \cdot \sum_{\epsilon^y \in \{\pm 1\}^m} \theta(\epsilon^y)^2 \det\big(\pi_1^{(m)}(y_1) \quad \ldots \quad \pi_1^{(m)}(y_m)\big) \\ &= 2^{\hat{m}} \det\big(\pi_0^{(\hat{m})}(x_1) \quad \ldots \quad \pi_0^{(\hat{m})}(x_{\hat{m}})\big) \cdot 2^m \det\big(\pi_1^{(m)}(y_1) \quad \ldots \quad \pi_1^{(m)}(y_m)\big). \end{aligned}$$

By noting $\hat{m} + m = n$ and by expressing the result in terms of Vandermonde determinants, we finally get

$$D(x;y) = 2^n \cdot \Delta(x_1^2, \ldots, x_{\hat{m}}^2) \cdot y_1 \cdots y_m \, \Delta(y_1^2, \ldots, y_m^2). \tag{2.7}$$

This factorization establishes Theorem 2.1.

Proof by Polynomiality

We give a second proof of the factorization (2.7) based on the observation that the sum in (2.5) defines a polynomial of homogeneous degree at most $\binom{n}{2}$. We identify the factors by symmetrizing known vanishings of this polynomial. Extending the definition of D polynomially to all real values of its arguments, we get in particular

$$D(\epsilon_1\sigma_1, \ldots, \epsilon_n\sigma_n) = \theta_0(\epsilon) D(\sigma_1, \ldots, \sigma_n) \quad (\epsilon \in \{\pm 1\}^n) \tag{2.8}$$

and, inherited from the Vandermonde terms, D is *antisymmetric* with respect to permutations of either the even or odd indices of σ since both sets of permutations leave the factor $\theta_0(\epsilon)$ invariant.

Now, if $\sigma_j = \sigma_{j+2}$ for σ belonging to the cone (2.2), we have $\sigma_j = \sigma_{j+1} = \sigma_{j+2}$ and, hence, by the pigeonhole principle, for each choice of signs ϵ at least one of

$$\epsilon_j\sigma_j = \epsilon_{j+1}\sigma_{j+1} \quad \text{or} \quad \epsilon_j\sigma_j = \epsilon_{j+2}\sigma_{j+2} \quad \text{or} \quad \epsilon_{j+1}\sigma_{j+1} = \epsilon_{j+2}\sigma_{j+2}$$

holds. Therefore, each of the Vandermonde terms in (2.5) vanishes. It follows that $\sigma_j - \sigma_{j+2}$ and by (2.8) also $\sigma_j + \sigma_{j+2}$ divide D for all j, thus so does the product $\sigma_j^2 - \sigma_{j+2}^2$. We also note that if $\sigma_2 = 0$, we have $\sigma_1 = \sigma_2 = 0$ and, hence, for each choice of signs $\epsilon_1\sigma_1 = \epsilon_2\sigma_2$. Once more each of the Vandermonde terms in (2.5) vanishes and it follows that σ_2 divides D.

In terms of the (x, y)-coordinates (2.3a) we thus see that $x_j^2 - x_{j+1}^2$, $y_j^2 - y_{j+1}^2$ and y_1 divide $D(x; y)$. Invoking the antisymmetry with respect to either x or y, we see that D is divisible by $x_j^2 - x_k^2$, $y_j^2 - y_k^2$ for every $j \neq k$ and by y_j for every j. These factors contribute homogeneous degree

$$2\binom{\hat{m}}{2} + 2\binom{m}{2} + m = \binom{n}{2},$$

so D cannot have any other non-unit factors and we get

$$D(x; y) = d_n \cdot \Delta(x_1^2, \ldots, x_{\hat{m}}^2) \cdot y_1 \cdots y_m \, \Delta(y_1^2, \ldots, y_m^2) \tag{2.9}$$

with a positive constant d_n. This is (2.7) except for identifying $d_n = 2^n$.

Remark 2.2. The value of d_n can easily be calculated without resorting to the first proof: a straightforward inspection shows that the expressions (2.5) and (2.9) both induce an asymptotics of the form

$$D(\sigma_1, \ldots, \sigma_{n-1}, \sigma_n) \sim \kappa_n \sigma_n^{n-1} D(\sigma_1, \ldots, \sigma_{n-1}) \quad (\sigma_n \to \infty);$$

the first one gives $\kappa_n = 2$, the second one $\kappa_n = d_n/d_{n-1}$. From $D(\sigma_1) = 2$ we thus get $d_n = 2^n$.

3. Singular Values of Bordered (Skew-)Symmetric Matrices

In preparation for what follows, in this section we study an algebraic device that allows us to untangle the interlacing (2.4b) of even and odd singular values, namely bordering a *skew-symmetric* matrix $A \in \mathbb{R}^{n\times n}$ with a column vector: $A \mapsto (b \;\; A)$.

By looking at the purely imaginary Hermitian matrix iA we see that each non-zero singular value of A occurs with even multiplicity. That is, with $n = 2m + \mu$, the n singular values of A can be arranged as the sequence $s_1, s_1, s_2, s_2, \ldots, s_m, s_m$ and, if $\mu = 1$, also $s_{m+1} = 0$, decreasingly ordered according to

$$s_1 \geq s_2 \geq \cdots \geq s_{\hat{m}}, \quad \hat{m} = m + \mu.$$

The results of this section are twofold. First, Lemma 3.1 shows that the singular values of $(b \ \ A)$ are of the form (with the value $s_{\hat{m}} = 0$ only formally added to the list of inequalities if $\mu = 1$)

$$t_1 \geq s_1 \geq t_2 \geq s_2 \geq \cdots \geq t_{\hat{m}} \geq s_{\hat{m}}. \tag{2.4b}$$

That is, bordering A by a column b splits the double listed pairs (s_j, s_j) of singular values into (s_j, t_j) and, if $\mu = 1$, modifies the surplus singular value $s_{\hat{m}} = 0$ into some $t_{\hat{m}}$ subject to the interlacing (2.4b). They are *strictly interlacing* if

$$t_1 > s_1 > t_2 > s_2 > \cdots > t_{\hat{m}} > s_{\hat{m}}.$$

Second, Lemma 3.2 establishes a coordinate change $(t, s) \mapsto (r, s)$ through an explicit algebraic map such that strict interlacing of t with s corresponds to strict positivity of the components of r.

To begin with, there are orthogonal matrices U and V such that (the last row and column of the block partitioning are understood to be μ-dimensional, meaning that they are missing if $\mu = 0$)

$$UAV' = \begin{pmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad U(b \ \ A)\begin{pmatrix} 1 & 0 \\ 0 & V' \end{pmatrix} = \begin{pmatrix} u & S & 0 & 0 \\ v & 0 & S & 0 \\ \eta & 0 & 0 & 0 \end{pmatrix}, \tag{3.1}$$

with $S = \operatorname{diag}(s_1, \ldots, s_m)$ built from the singular values of A and the partitioning

$$Ub = \begin{pmatrix} u \\ v \\ \eta \end{pmatrix}, \quad u, v \in \mathbb{R}^m, \ \ \eta \in \mathbb{R}^\mu.$$

Hence, the singular values of $(b \ \ A)$ are given by the following lemma.

Lemma 3.1. *Let* $S = \operatorname{diag}(s_1, \ldots, s_m)$ *be a diagonal matrix and* $u, v \in \mathbb{R}^m$, $\eta \in \mathbb{R}^\mu$ $(\mu = 0, 1)$. *Then, with* $\hat{m} = m + \mu$, *the singular values of the* $(m + \hat{m}) \times (m + \hat{m} + 1)$ *block matrix*

$$\begin{pmatrix} u & S & 0 & 0 \\ v & 0 & S & 0 \\ \eta & 0 & 0 & 0 \end{pmatrix}$$

are $s_1, \ldots, s_m, t_1, \ldots, t_{\hat{m}}$, *satisfying the interlacing property* (2.4b). *Here, the* t_j *are the singular values of the* $\hat{m} \times (\hat{m} + 1)$ *bordered matrix* $(r \ \ \hat{S})$ *with*

$\hat{S} = \operatorname{diag}(s_1, \ldots, s_{\hat{m}})$ and $r_j = \sqrt{u_j^2 + v_j^2}$ $(j = 1, \ldots, m)$; if $\mu = 1$, then $r_{m+1} = |\eta|$ and $s_{m+1} = 0$. Further, there holds

$$t_1^2 + \cdots + t_{\hat{m}}^2 = r_1^2 + \cdots + r_{\hat{m}}^2 + s_1^2 + \cdots + s_m^2 \tag{3.2}$$

and, if $\mu = 1$,

$$t_1 \cdots t_{\hat{m}} = r_{\hat{m}} \cdot s_1 \cdots s_m. \tag{3.3}$$

Proof. It suffices to prove that the two matrices

$$M_1 = \begin{pmatrix} u & S & 0 & 0 \\ v & 0 & S & 0 \\ \eta & 0 & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & S & 0 & 0 \\ r & 0 & S & 0 \\ |\eta| & 0 & 0 & 0 \end{pmatrix}$$

with $r_j = \sqrt{u_j^2 + v_j^2}$, $j = 1, \ldots, m$, have the same singular values. Using a Givens rotation U_j with

$$U_j \begin{pmatrix} u_j \\ v_j \end{pmatrix} = \begin{pmatrix} 0 \\ r_j \end{pmatrix} \quad (j = 1, \ldots, m)$$

one gets

$$U_j \begin{pmatrix} u_j & s_j & 0 \\ v_j & 0 & s_j \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & U_j' \end{pmatrix} = \begin{pmatrix} 0 & s_j & 0 \\ r_j & 0 & s_j \end{pmatrix}.$$

Hence, by successively applying these two-dimensional orthogonal operations to the corresponding rows and columns (and addressing a possible sign change of the last row if $\mu = 1$) one transforms M_1 into M_2 while leaving the singular values invariant.

We note that the decreasingly ordered singular values $s_1, \ldots, s_{\hat{m}}$ of a matrix $\hat{S}$ and those of the bordered matrix $(r \ \ \hat{S})$, $t_1, \ldots, t_{\hat{m}}$, are generally known [13, Corollary 7.3.6] to be *interlacing* as in (2.4b).

To finish, (3.2) follows from expressing the Frobenius norm of $(r \ \ \hat{S})$ in terms of its singular values t and (3.3) follows from doing the same, if $\mu = 1$, for the magnitude of the determinant of that matrix with the last column (which is all zeros then) deleted. □

The next lemma shows that one can uniquely solve the inverse problem $t \mapsto r$ for strict interlacing.

Lemma 3.2. *With the notation as in Lemma 3.1, let $s_1 > s_2 > \cdots > s_{\hat{m}} \geq 0$ with $s_{\hat{m}} = 0$ if $\mu = 1$, and let $\hat{S} = \operatorname{diag}(s_1, \ldots, s_{\hat{m}})$. Then, the map*

$$\Phi : r \mapsto t = \text{the decreasingly ordered singular values of } (r \ \ \hat{S})$$

defines a diffeomorphism

$$\Phi : \mathbb{R}_{>0}^{\hat{m}} \to \{t \in \mathbb{R}_{>0}^{\hat{m}} : t \text{ is strictly interlacing with } s\}.$$

If t is strictly interlacing with s, its preimage $r = \Phi^{-1}(t)$ is the unique positive solution of the system

$$\sum_{k=1}^{\hat{m}} \frac{r_k^2}{t_j^2 - s_k^2} = 1 \quad (j = 1, \ldots, \hat{m}), \tag{3.4}$$

which is explicitly solved by

$$r_j^2 = -\frac{\omega_t(s_j^2)}{\omega_s'(s_j^2)}, \quad \omega_s(\xi) = (\xi - s_1^2) \cdots (\xi - s_{\hat{m}}^2), \quad \omega_t(\xi) = (\xi - t_1^2) \cdots (\xi - t_{\hat{m}}^2). \tag{3.5}$$

The Jacobian of the inverse map Φ^{-1} is given by

$$\det\left(\frac{\partial r_j}{\partial t_k}\right)_{1 \le j,k \le \hat{m}} = \frac{1}{r_1 \cdots r_m} \cdot \frac{(t_1 \cdots t_{\hat{m}})^{1-\mu}\, \Delta(t_{\hat{m}}^2, \ldots, t_1^2)}{(s_1 \cdots s_m)^{\mu}\, \Delta(s_m^2, \ldots, s_1^2)} \quad (\mu = 0, 1). \tag{3.6}$$

Proof. The squares of the singular values $t_1, \ldots, t_{\hat{m}}$ of $(r \ \hat{S})$ are the eigenvalues of

$$(r \ \ \hat{S})(r \ \ \hat{S})' = \hat{S}\hat{S}' + rr' = \operatorname{diag}(s_1^2, \ldots, s_{\hat{m}}^2) + rr'.$$

Now, any set of values t_j^2 for which t_j satisfies the interlacing property (2.4b) can be obtained in this way, that is, as the eigenvalues of a positive semi-definite rank-one perturbation of $D = \operatorname{diag}(s_1^2, \ldots, s_{\hat{m}}^2)$ (see, e.g. [18, Sec. 2]). Since rr' does not depend on the signs of the individual entries of r, we can always choose $r \in \mathbb{R}^{\hat{m}}_{\ge 0}$. If $r_\nu = 0$ for some ν, then the νth row and the νth column of rr' are zero which means that s_ν^2 appears among the values of t_j^2. Hence, *strict* interlacing implies $r \in \mathbb{R}^{\hat{m}}_{>0}$.

Given such an $r \in \mathbb{R}^{\hat{m}}_{>0}$, the eigenvalues t_j^2 of $D + rr'$ are known (see [12, Lemma 8.4.3]) to be *strictly* interlacing with the s_k^2 and satisfy the secular equation

$$f(t_j^2) = 0 \quad (j = 1, \ldots, \hat{m}), \quad f(\lambda) = 1 + r'(D - \lambda I)^{-1} r,$$

which is (3.4). Since the determinant (3.7) given below is non-zero and, hence, the Cauchy matrix

$$C = \left(\frac{1}{t_j^2 - s_k^2}\right)_{1 \le j,k \le \hat{m}}$$

is non-singular, there is a one-to-one correspondence of $r \in \mathbb{R}^{\hat{m}}_{>0}$ with those t that *strictly* interlace with s. Because each of the steps $t \mapsto C \mapsto r$ is smooth, we have therefore proved that Φ is a diffeomorphism.

By relating Cauchy matrices with Lagrangian polynomial interpolation, Schechter [16, Eq. (16)] gave a short and simple proof of the explicit formula (3.5). Differentiation with respect to t_k gives

$$J_{jk} = \frac{\partial r_j}{\partial t_k} = \frac{r_j t_k}{s_j^2 - t_k^2}.$$

Hence,

$$J = \operatorname{diag}(r_1, \dots, r_{\hat{m}}) C \operatorname{diag}(t_1, \dots, t_{\hat{m}}),$$

which implies $\det J = t_1 \cdots t_{\hat{m}}\, r_1 \cdots r_{\hat{m}}\, \det C$. Now, using the explicit determinantal formula [16, Eq. (4)]

$$\det C = \frac{\prod_{j<k}(t_j^2 - t_k^2)(s_k^2 - s_j^2)}{\prod_{j,k}(t_j^2 - s_k^2)} \tag{3.7}$$

and

$$r_1^2 \cdots r_{\hat{m}}^2 = (-1)^{\hat{m}} \frac{\omega_t(s_1^2) \cdots \omega_t(s_{\hat{m}}^2)}{\omega_s'(s_1^2) \cdots \omega_s'(s_{\hat{m}}^2)},$$

together with the following straightforward evaluations of the product terms

$$(-1)^{\hat{m}} \omega_t(s_1^2) \cdots \omega_t(s_{\hat{m}}^2) = \prod_{j,k}(t_j^2 - s_k^2), \quad \omega_s'(s_1^2) \cdots \omega_s'(s_{\hat{m}}^2) = \prod_{j \neq k}(s_j^2 - s_k^2),$$

one gets

$$\det J = \frac{t_1 \cdots t_{\hat{m}}}{r_1 \cdots r_{\hat{m}}} \cdot \frac{\prod_{j<k}(t_j^2 - t_k^2)}{\prod_{j<k}(s_j^2 - s_k^2)}.$$

With

$$\prod_{j<k}(t_j^2 - t_k^2) = \Delta(t_{\hat{m}}^2, \dots, t_1^2),$$

$$\prod_{j<k}(s_j^2 - s_k^2) = \Delta(s_{\hat{m}}^2, \dots, s_1^2) = (s_1 \cdots s_m)^{2\mu} \Delta(s_m^2, \dots, s_1^2),$$

one finally gets the expression (3.6) by using (3.3) if $\mu = 1$. □

4. Random Matrix Models for the Odd and Even Singular Values

Because of interlacing, the factorization of the joint density stated in Theorem 2.1 does not reveal an independence between the x and the y components of the singular values, or to the same end, between the t and the s components. If we change, however, the (t, s) coordinates to the (r, s) coordinates introduced in Lemma 3.2, the interlacing is replaced by just a positivity condition on the r components. The following theorem, which sharpens Theorem 1.1, shows that not only are the r and the s components independent of each other but both sets of components have so much additional structure that they can be completely described in terms of known distributions.

Theorem 4.1. *Applying the transform* $(t, s) \mapsto (r, s)$ *of Lemma* 3.2 *to the* (t, s) *parametrization* (2.4) *of the decimated ensembles* $\operatorname{odd}|\mathrm{GOE}_n|$ *and* $\operatorname{even}|\mathrm{GOE}_n|$ *defines a set of random variables* r_k*, which are distributed as* χ_2 *for* $k = 1, \dots, m$

and, if $\mu = 1$, *distributed as* χ_1 *for* $k = m+1$. *They are independent of each other and the even singular values* s, *which are jointly distributed as*

$$\text{even}|\text{GOE}_n| \stackrel{\text{d}}{=} \text{aGUE}_n.$$

Proof. In terms of the (t,s) coordinates, the joint density (2.6) of the singular values of the GOE can be recast in the form

$$\begin{aligned} q(s;t) &= c_n 2^n n! \cdot (s_1 \cdots s_m)^\mu \, (t_1 \cdots t_{\hat m})^{1-\mu} \\ &\quad \cdot \Delta(s_m^2, \ldots, s_1^2)\Delta(t_{\hat m}^2, \ldots, t_1^2) \cdot e^{-\sum_{j=1}^m \frac{s_j^2}{2} - \sum_{j=1}^{\hat m} \frac{t_j^2}{2}}, \end{aligned} \tag{4.1}$$

where the case distinction between even ($\mu = 0$) and odd ($\mu = 1$) orders n has been expressed in terms of powers. If we apply the coordinate change $(t,s) \mapsto (r,s)$ of Lemma 3.2, which is a diffeomorphism up to an exceptional set of zero probability, the density with respect to the (r,s) is

$$\begin{aligned} &\left(\det\left(\frac{\partial r_j}{\partial t_k}\right)_{1\le j,k\le \hat m}\right)^{-1} \cdot q(s;t) \\ &\quad = r_1 \cdots r_m \cdot \frac{(s_1\cdots s_m)^\mu \, \Delta(s_m^2,\ldots,s_1^2)}{(t_1\cdots t_{\hat m})^{1-\mu}\,\Delta(t_{\hat m}^2,\ldots,t_1^2)} \; q(s;t) \\ &\quad = \left(\prod_{j=1}^m r_j e^{-r_j^2/2}\right) \cdot \left(\sqrt{\frac{2}{\pi}}\, e^{-r_{\hat m}^2/2}\right)^{\mu} \cdot \left(\delta_\mu c_n 2^n n! \cdot \prod_{j=1}^m s_j^{2\mu} e^{-s_j^2} \cdot \Delta(s_m^2,\ldots,s_1^2)^2\right) \end{aligned}$$

with $\delta_\mu = (\pi/2)^{\mu/2}$. Here we used expression (3.6) for the Jacobian and simplified the exponential functions according to (3.2). On their supporting domains, the first m factors of the resulting density are a χ_2-density each, the next one is a χ_1-density if $\mu = 1$ (disappearing if $\mu = 0$), and the last one is the joint density of the anti-GUE of order n, see [15, Sec. 13.1] or [9, Exercise 1.3.5(iv)]. ☐

Remark 4.1. As a side product, the proof shows that the normalization constant a_n of the joint density of the anti-GUE, if extended by symmetry to be supported on $[0,\infty)^m$, is given by

$$a_n = c_n \left(\frac{\pi}{2}\right)^{\mu/2} \frac{2^n n!}{m!} \quad (n = 2m+\mu, \mu = 0,1).$$

This is consistent with the explicit formulae for c_n^{-1} and a_n^{-1} given in [9, Eqs. (1.163) and (4.157)].

The proof of Theorem 4.1 shows that the joint density $p(t\,|\,s)$ of the t variables conditioned on s is given by the expression

$$p(t\,|\,s) = \frac{1}{\delta_\mu} \frac{(t_1\cdots t_{\hat m})^{1-\mu}\,\Delta(t_{\hat m}^2,\ldots,t_1^2)}{(s_1\cdots s_m)^\mu\,\Delta(s_m^2,\ldots,s_1^2)} e^{-\sum_{j=1}^{\hat m} t_j^2/2 + \sum_{j=1}^m s_j^2/2}. \tag{4.2}$$

This is just a particular case of a general result by Forrester and Rains [11, Corollary 3], which gives the probability $p(t\,|\,s)$ if the t_k are the solutions of the secular equation (3.4) with parameters r_j being independently gamma distributed. In retrospect, we could thus have proved Theorem 4.1, based on Theorem 2.1, starting with the Forrester–Rains formula (4.2) and working backwards.

Now, Theorem 4.1 and Lemma 3.1 yield a new random matrix model for $|\mathrm{GOE}_n|$ which amounts to the singular values of certain bordered skew-symmetric Gaussian matrices.

Corollary 4.1. *Let $X \in \mathbb{R}^{n\times n}$ be a random matrix with independent standard normal entries. Denote by $G = (X+X')/2$ its symmetric and by $A = (X-X')/2$ its skew-symmetric part. Let τ_n be a χ_n-distributed random variable that is independent of X. Then, both the singular values of G and the singular values of the bordered matrix (with e_1 denoting the first unit vector)*

$$H = (\tau_n e_1 \;\; A) \tag{4.3}$$

are jointly distributed as those of the GOE *of order n. The same holds if the matrix H is obtained from bordering A with an independent standard normal vector.*

Proof. Note that the singular values of the symmetric part $G = (X + X')/2$ are by *definition* jointly distributed as the singular values of the GOE of order n.

As discussed in the derivation of (3.1), the singular value decomposition of A takes the form

$$UAV' = \begin{pmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad S = \mathrm{diag}(s_1, \ldots, s_m),$$

where the last row and column are missing if n is odd. By symmetry, the orthogonal matrix U is Haar distributed — independently of $\{s_1, \ldots, s_m\}$, which are jointly distributed as the anti-GUE of order n, cf. [15, Sec. 13.1] or [9, Example 1.3.5(iv)]. Applying U to the first column H_1 of H defines

$$UH_1 = \tau_n U_1 = \begin{pmatrix} u \\ v \\ \eta \end{pmatrix}, \quad u, v \in \mathbb{R}^m, \;\; \eta \in \mathbb{R}^\mu.$$

Since the first column U_1 of U is uniformly distributed on the sphere S^{n-1} and τ_n is independently χ_n-distributed, we see that UH_1 is a standard normal vector, see, e.g. [5, Sec. V.4]; the same conclusion holds for standard normal H_1. Hence, the variables

$$r_j = \sqrt{u_j^2 + v_j^2} \quad (j = 1, \ldots, m)$$

are independently χ_2-distributed and, if $\mu = 1$, $r_{m+1} = |\eta|$ is independently χ_1-distributed. Comparing the results of Lemma 3.1 and Theorem 4.1 finishes the proof. $\square$

The random matrix model of the last corollary can easily be turned into the following sparse model that separates the even and odd singular values. Note that just one of the two matrices is square, the other is rectangular.

Corollary 4.2. *Let $n = 2m + \mu$, $\mu = 0, 1$ and let $\tau_1, \ldots, \tau_n$ be independent random variables, with τ_k distributed as χ_k. The union of the singular values of both the bidiagonal matrix*

$$B_\mu^{\text{odd}} = (\tau_n e_1 \;\; B_\mu^{\text{even}}) \in \mathbb{R}^{(m+\mu)\times(m+1)}$$

and the bidiagonal matrix $B_\mu^{\text{even}} \in \mathbb{R}^{(m+\mu)\times m}$, defined by

$$B_0^{\text{even}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tau_{2m-1} & & & \\ \tau_{2m-2} & \tau_{2m-3} & & \\ & \ddots & \ddots & \\ & & \tau_2 & \tau_1 \end{pmatrix},$$

$$B_1^{\text{even}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tau_{2m} & & & \\ \tau_{2m-1} & \tau_{2m-2} & & \\ & \ddots & \ddots & \\ & & \tau_3 & \tau_2 \\ & & & \tau_1 \end{pmatrix},$$

is jointly distributed as $|\mathrm{GOE}_n|$. Here, the singular values of B_μ^{odd} correspond to $\mathrm{odd}|\mathrm{GOE}_n|$ *and the singular values of B_μ^{even} correspond to* $\mathrm{even}|\mathrm{GOE}_n|$, *both drawn from the same ensemble.*

Proof. Using the notation of Corollary 4.1, a Householder tridiagonalization of A, rescaling rows and columns by -1 as necessary, yields $UAV' = T$ with orthogonal matrices U, V and

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \tau_{n-1} & & & & \\ \tau_{n-1} & 0 & \tau_{n-2} & & & \\ & \tau_{n-2} & 0 & \tau_{n-3} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \tau_2 & 0 & \tau_1 \\ & & & & \tau_1 & 0 \end{pmatrix},$$

where the entries τ_k are jointly distributed as independent χ_k-variables with degrees of freedom $k = 1, \ldots, n$, see [6, Sec. II]. Since Householder triadiagonalizations do not operate on the first row and column, we have $Ue_1 = e_1$ and, hence,

$$U \cdot (\tau_n e_1 \;\; A) \cdot \begin{pmatrix} 1 & 0 \\ 0 & V' \end{pmatrix} = (\tau_n e_1 \;\; T).$$

Thus the matrices $(\tau_n e_1 \;\; A)$ and $(\tau_n e_1 \;\; T)$ have the same singular values. By a simultaneous row and column permutation of T, the odd columns and rows occur before the even ones, we see that the matrices $(\tau_n e_1 \;\; T)$ and (with the length of the first unit vector e_1 adjusted)

$$\begin{pmatrix} \tau_n e_1 & 0 & B_\mu^{\text{even}} \\ 0 & (B_\mu^{\text{even}})' & 0 \end{pmatrix}$$

have the same singular values. Interlacing shows that the singular values of B_μ^{even} correspond to the even ones of $(\tau_n e_1 \;\; A)$ and the singular values of $(\tau_n e_1 \;\; B_\mu^{\text{even}})$ correspond to the odd ones. □

Remark 4.2. The mapping from the singular values of a sample drawn from the GOE to τ coordinates is deterministic and can be made explicit (the same remark applies to the construction of the ξ variables in the next section): starting with (r, s) coordinates, the τ are obtained by applying appropriate orthogonal row and column transformations to the matrix

$$\begin{pmatrix} 0 & 0 & S' \\ r & -S & 0 \end{pmatrix}, \quad \text{where } S = \begin{pmatrix} s_1 & & \\ & \ddots & \\ & & s_m \\ 0 & \cdots & 0 \end{pmatrix};$$

if $\mu = 0$, the last row of S is missing.

5. Square Bidiagonal Matrix Models and the Determinant

To study the distribution of determinants we turn the bidiagonal random matrix model of Corollary 4.2 into one with square matrices only. Key to this transformation is the following variant of a result by Dumitriu and Forrester [6, Claim 6.5].

Lemma 5.1. *Let the variables τ_k $(k = 1, \ldots, 2m-1)$ be distributed as χ_k, with the distribution of τ_{2m} arbitrary such that $\tau_1, \ldots, \tau_{2m}$ are independent of each other. Then the singular values of the $m \times (m+1)$ bidiagonal matrix*

$$B = \begin{pmatrix} \tau_{2m} & \tau_{2m-1} & & & \\ & \tau_{2m-2} & \tau_{2m-3} & & \\ & & \ddots & \ddots & \\ & & & \tau_2 & \tau_1 \end{pmatrix} \tag{5.1}$$

are the same as those of the $m \times m$ bidiagonal matrix

$$R = \begin{pmatrix} \xi_{2m+1} & \xi_{2m-2} & & & \\ & \xi_{2m-1} & \xi_{2m-4} & & \\ & & \ddots & \ddots & \\ & & & \xi_5 & \xi_2 \\ & & & & \xi_3 \end{pmatrix} \tag{5.2}$$

constructed by the normalized reduced RQ-decomposition[b] $B = RQ$ *with a row-orthogonal matrix Q, that is, by the almost surely positive solution of the set of equations*

$$\xi_{2k+1}^2 + \xi_{2k-2}^2 = \tau_{2k}^2 + \tau_{2k-1}^2 \quad (k = 1, \ldots, m), \tag{5.3a}$$

$$\xi_{2k+1}\xi_{2k} = \tau_{2k+1}\tau_{2k} \qquad (k = 1, \ldots, m-1). \tag{5.3b}$$

The variables $\xi_2, \ldots, \xi_{2m-1}$ are distributed as $\chi_2, \ldots, \chi_{2m-1}$; they are independent of each other and τ_{2m}. The variable ξ_{2m+1} is of the form

$$\xi_{2m+1} = \sqrt{\xi_1^2 + \tau_{2m}^2}, \quad \textit{where } \xi_1 = \sqrt{\tau_{2m-1}^2 - \xi_{2m-2}^2}$$

is distributed as χ_1 and is also independent of $\xi_2, \ldots, \xi_{2m-1}$ and τ_{2m}.

Proof. A well-known result [5, Theorem IX.3.1] about the χ^2-distribution states that the involution

$$\phi(X, Y, Z) = \left(Z\frac{X}{X+Y}, Z\frac{Y}{X+Y}, X+Y \right) \tag{5.4}$$

maps a set of mutually independent random variables X, Y, Z distributed as χ_r^2, χ_s^2 and χ_{r+s}^2 to a new set of mutually independent random variables of exactly the same type. Starting with $\tau_{1,1} = \tau_1$, the system (5.3) is recursively solved for the variables $\xi_2, \ldots, \xi_{2m-1}$ by

$$\left(\tau_{1,k+1}^2, \xi_{2k}^2, \xi_{2k+1}^2\right) = \phi\left(\tau_{1,k}^2, \tau_{2k}^2, \tau_{2k+1}^2\right), \quad (k = 1, \ldots, m-1).$$

Hence, the variable $\xi_1 = \tau_{1,m}$ and the thus constructed $\xi_2, \ldots, \xi_{2m-1}$ are independent of each other and the not yet used variable τ_{2m}; they are distributed as χ_k $(k = 1, \ldots, 2m-1)$. Because ϕ is an involution, there is

$$\tau_{2m-1}^2 = \tau_{1,m}^2 + \xi_{2m-2}^2 = \xi_1^2 + \xi_{2m-2}^2.$$

Hence, the yet to be used $k = m$ case of Eq. (5.3a) finally implies the asserted form of ξ_{2m+1}. □

[b]The R-factor can equivalently be obtained from the Cholesky-type decomposition $RR' = BB'$.

Remark 5.1. The use of the involution (5.4) has been motivated by the observation [7, Lemma 1] that for 2×2 matrices the R-factor of the RQ-decomposition of a lower triangular matrix

$$L = \begin{pmatrix} z & 0 \\ y & x \end{pmatrix} = RQ, \quad R = \begin{pmatrix} \xi & \eta \\ 0 & \zeta \end{pmatrix},$$

is induced by the transformation $(\xi^2, \eta^2, \zeta^2) = \phi(x^2, y^2, z^2)$. Basically, the R-factor (5.2) of the bidiagonal matrix (5.1) is then obtained by successively applying this transformation along the diagonal from the lower right to the upper left.

Application of this lemma to Corollary 4.2 yields sparse random matrix models for the odd and even singular values of the GOE in terms of bidiagonal *square* matrices. We begin with the case of even order $n = 2m$.

Theorem 5.1. *Let $\xi_1, \ldots, \xi_{2m}$ be independent random variables, with ξ_k distributed as χ_k. The union of the singular values of the two bidiagonal square matrices*

$$R_0^{\text{odd}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\xi_1^2 + 2\xi_{2m}^2} & \xi_{2m-2} & & & \\ & \xi_{2m-1} & \xi_{2m-4} & & \\ & & \ddots & \ddots & \\ & & & \xi_5 & \xi_2 \\ & & & & \xi_3 \end{pmatrix}$$

and

$$R_0^{\text{even}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_1 & \xi_{2m-2} & & & \\ & \xi_{2m-1} & \xi_{2m-4} & & \\ & & \ddots & \ddots & \\ & & & \xi_5 & \xi_2 \\ & & & & \xi_3 \end{pmatrix}$$

is jointly distributed as $|\mathrm{GOE}_{2m}|$. *Here, the singular values of R_0^{odd} correspond to* $\mathrm{odd}|\mathrm{GOE}_{2m}|$ *and the singular values of R_0^{even} correspond to* $\mathrm{even}|\mathrm{GOE}_{2m}|$, *both drawn from the same ensemble.*

Proof. Let $\tau_1, \ldots, \tau_{2m}$ be independent random variables, with τ_k distributed as χ_k. Prepending a zero column to the second matrix, Corollary 4.2 shows that the singular values of the two matrices

$$B_0^{\text{odd}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tau_{2m}^{\text{odd}} & \tau_{2m-1} & & & \\ & \tau_{2m-2} & \tau_{2m-3} & & \\ & & \ddots & \ddots & \\ & & & \tau_2 & \tau_1 \end{pmatrix}$$

and

$$B_0^{\text{even}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tau_{2m}^{\text{even}} & \tau_{2m-1} & & & \\ & \tau_{2m-2} & \tau_{2m-3} & & \\ & & \ddots & \ddots & \\ & & & \tau_2 & \tau_1 \end{pmatrix},$$

where

$$\tau_{2m}^{\text{odd}} = \sqrt{2}\tau_{2m}, \quad \tau_{2m}^{\text{even}} = 0,$$

are jointly distributed as $|\text{GOE}_{2m}|$. Here, the singular values of B_0^{odd} correspond to odd$|\text{GOE}_{2m}|$ and the singular values of B_0^{even} correspond to even$|\text{GOE}_{2m}|$, both drawn from the same ensemble. If we apply the construction of Lemma 5.1 to both matrices B_0^{odd} and B_0^{even} simultaneously, we obtain the R-factors R_0^{odd} and R_0^{even} with one and the same set of variables $\xi_1, \ldots, \xi_{2m}$ subject to the asserted properties and additionally

$$\xi_{2m+1}^{\text{odd}} = \sqrt{\xi_1^2 + (\tau_{2m}^{\text{odd}})^2} = \sqrt{\xi_1^2 + 2\tau_{2m}^2},$$

$$\xi_{2m+1}^{\text{even}} = \sqrt{\xi_1^2 + (\tau_{2m}^{\text{even}})^2} = \xi_1.$$

By defining $\xi_{2m} = \tau_{2m}$ we thus get the asserted form of R_0^{odd} and R_0^{even}. □

Remark 5.2. The matrices R_0^{even} of Theorem 5.1 and B_0^{even} of Corollary 4.2 are superficially related, up to a different sample of the independent random variables, by a transposition followed by a cyclic permutation of their diagonals. Such a transformation would not, in general, preserve singular values, but depends instead on properties of the χ distributions.

The case of odd order exhibits a similar structure. The equivalence between the models B_1^{even} and R_1^{even} used in the following theorem is also noted, for the anti-GUE, in [7, Sec. 2; 6, Claim 6.5].

Theorem 5.2. *Let $\xi_1, \ldots, \xi_{2m+1}$ be independent random variables, with ξ_k distributed as χ_k. The union of the singular values of the two bidiagonal square matrices*

$$R_1^{\text{odd}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2}\xi_1 & \sqrt{2}\xi_{2m} & & & \\ & \xi_{2m+1} & \xi_{2m-2} & & \\ & & \ddots & \ddots & \\ & & & \xi_5 & \xi_2 \\ & & & & \xi_3 \end{pmatrix}$$

and

$$R_1^{\text{even}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_{2m+1} & \xi_{2m-2} & & & \\ & \xi_{2m-1} & \xi_{2m-4} & & \\ & & \ddots & \ddots & \\ & & & \xi_5 & \xi_2 \\ & & & & \xi_3 \end{pmatrix}$$

is jointly distributed as $|\mathrm{GOE}_{2m+1}|$. *Here, the singular values of* R_1^{odd} *correspond to* $\mathrm{odd}|\mathrm{GOE}_{2m+1}|$ *and the singular values of* R_1^{even} *correspond to* $\mathrm{even}|\mathrm{GOE}_{2m+1}|$, *both drawn from the same ensemble.*

Proof. Let $\tau_1, \ldots, \tau_{2m+1}$ be independent random variables, with τ_k distributed as χ_k. By transposing both matrices and prepending a zero column to the first one, Corollary 4.2 shows that the singular values of the two matrices

$$B_1^{\text{odd}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \sqrt{2}\tau_{2m+1} & & & \\ & \tau_{2m} & \tau_{2m-1} & & \\ & & \ddots & \ddots & \\ & & & \tau_2 & \tau_1 \end{pmatrix}$$

and

$$B_1^{\text{even}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tau_{2m} & \tau_{2m-1} & & & \\ & \tau_{2m-2} & \tau_{2m-3} & & \\ & & \ddots & \ddots & \\ & & & \tau_2 & \tau_1 \end{pmatrix}$$

are jointly distributed as $|\mathrm{GOE}_{2m+1}|$. Here, the singular values of B_0^{odd} correspond to $\mathrm{odd}|\mathrm{GOE}_{2m+1}|$ and the singular values of B_0^{even} correspond to $\mathrm{even}|\mathrm{GOE}_{2m+1}|$, both drawn from the same ensemble. If we apply the construction of Lemma 5.1 simultaneously to B_1^{even} and

$$\hat{B}_1^{\text{odd}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \tau_{2m+2}^{\text{odd}} & \tau_{2m+1} & & & \\ & \tau_{2m} & \tau_{2m-1} & & \\ & & \ddots & \ddots & \\ & & & \tau_2 & \tau_1 \end{pmatrix}, \quad \tau_{2m+2}^{\text{odd}} = 0,$$

which is B_1^{odd} with its first row rescaled, we obtain the R-factors R_1^{even} and

$$\hat{R}_1^{\text{odd}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_{2m+3}^{\text{odd}} & \xi_{2m} & & & \\ & \xi_{2m+1} & \xi_{2m-2} & & \\ & & \ddots & \ddots & \\ & & & \xi_5 & \xi_2 \\ & & & & \xi_3 \end{pmatrix}$$

with one and the same set of variables $\xi_1, \ldots, \xi_{2m+1}$ subject to the asserted properties and additionally

$$\xi_{2m+3}^{\mathrm{odd}} = \sqrt{\xi_1^2 + \left(\tau_{2m+2}^{\mathrm{odd}}\right)^2} = \xi_1.$$

By restoring the proper scaling of the first row, we thus get the asserted form of R_1^{even} and R_1^{odd}. □

Remark 5.3. This theorem implies that, for odd order $n = 2m + 1$, the product $\det R_1^{\mathrm{odd}}$ of the odd singular values of GOE_n and the product $\det R_1^{\mathrm{even}}$ of the even ones are related by

$$\det R_1^{\mathrm{odd}} = \xi_1 \det R_1^{\mathrm{even}},$$

where ξ_1 is a random variable distributed as χ_1 which is independent of $\mathrm{even}|\mathrm{GOE}_n|$. This is nothing but (3.3), recalling that by the proof of Theorem 4.1 the variable r_{m+1} is distributed as χ_1.

As an immediate consequence of the preceding theorems, stated in the following corollary, the magnitude of the determinant of the GOE can be expressed as a product of independent random variables. Even though Tao and Vu speculated that such a representation does not seem to be possible [17, p. 78], a precursor of this result was recognized implicitly by Delannay and Le Caër in [4, p. 1531] that the Meijer G-function they used to describe the distribution of the determinant when n is odd could be sampled as a product of independent gamma-distributed random variables. They did not, however, have any interpretation for these variables in terms of the underlying ensemble, nor did they recognize the possibility of sampling when n is even.

Corollary 5.1. *Let G_n be drawn from the* GOE *of order $n = 2m + \mu$ with parity $\mu = 0, 1$, $\hat{m} = m + \mu$. Then the determinant of $M_n = \sqrt{2} G_n$ factors into independent random variables of the form*

$$|\det M_n| = \eta_n^{(1)} \cdot \xi_3^2 \cdot \xi_5^2 \cdots \xi_{2\hat{m}-1}^2, \tag{5.5}$$

with

$$\eta_n^{(1)} = \xi_1 \cdot \sqrt{\xi_1^2 + 2\xi_n^2} \quad (\mu = 0), \qquad \eta_n^{(1)} = \sqrt{2}\, \xi_1 \quad (\mu = 1).$$

Here, the ξ_k are mutually independent random variables distributed as χ_k.

Proof. The assertion follows from the observation that $|\det M_n|$ is the product of the singular values of M_n and therefore, by Theorems 5.1 and 5.2, distributed as

$$\det(\sqrt{2} R_\mu^{\mathrm{even}}) \cdot \det(\sqrt{2} R_\mu^{\mathrm{odd}}).$$

Multiplication of the diagonal terms of the bidiagonal factors finishes the proof.

□

In retrospect, once we know that $|\det \mathrm{GOE}_n|$ is distributed as a product of $\hat{m}$ independent random variables, all the factors can be readily identified in the Mellin transforms computed by Delannay and Le Caër in [4, Eqs. (26) and (41)], although, for even $n = 2m$, the Mellin transform of the factor

$$\eta_{2m}^{(1)} = \xi_1 \sqrt{\xi_1^2 + 2\xi_{2m}^2}$$

into the expression

$$2^{3(s-1)/2} \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(s+m-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{s}{2}+m\right)} \, {}_2F_1\left(\frac{s}{2}, \frac{1-s}{2}; \frac{s}{2}+m; \frac{1}{2}\right) \tag{5.6}$$

with the hypergeometric function ${}_2F_1$ may not be familiar to most observers. This aspect of their paper has been missed by several commentators, with Mehta, in [15, Sec. 26.6], omitting their expression for even $n = 2m$ since the inverse Mellin transform of (5.6) cannot be readily written down.

Remark 5.4. In the case of odd order $n = 2m + 1$, the density of $\det \mathrm{GOE}_{2m+1}$ is necessarily odd, since the eigenvalue density (2.1) is even, but $\det(G) = -\det(-G)$. It follows that in this case the sign of the determinant is statistically independent of its magnitude, and we can obtain the distribution of the determinant by replacing ξ_1 by a standard normal variable. No corresponding result is available for even order $n = 2m$, although the factored presentation of the odd moments of $\det \mathrm{GOE}_{2m}$ in [1, Eq. (23)] suggests that the distribution of the determinant should involve many of the same factors.

6. Central Limit Theorem for the Determinant

Delannay and Le Caër used an explicit computation of the Mellin transform of the even part of the distribution of $\det \mathrm{GOE}_n$ to derive the cumulants of the potential $V = \log|\det \mathrm{GOE}_n|$, and to show that V is asymptotically Gaussian [4, Sec. III]. Tao and Vu extended this log-normality to determinants of a wider class of Wigner matrices, and provided an alternate proof in the case of Gaussian matrices in [17]: based on analyzing tridiagonal sparse models for the GOE and GUE eigenvalues, they found a way to approximate the log-determinant as a sum of *weakly dependent* terms, which then yields the asymptotic log-normality by stochastic calculus and the martingale central limit theorem. In this section we present yet another, much simpler proof based on the factorization of the magnitude of the determinant into independent random variables. In particular, our proof elucidates the difference between the GOE and the GUE in the scaling of the central limit theorem.

We start by recalling that, parallel to the factorization given in Corollary 5.1 for the GOE ($\beta = 1$), Edelman and La Croix [7, Theorem 2] obtained a factorization for the GUE ($\beta = 2$): with G_n drawn from the GUE of order n, the determinant of $M_n = \sqrt{2}G_n$ factors into independent random variables of the form

$$|\det M_n| = \eta_n^{(2)} \cdot \xi_3\tilde{\xi}_3 \cdot \xi_5\tilde{\xi}_5 \cdots \xi_{2\hat{m}-1}\tilde{\xi}_{2\hat{m}-1} \tag{6.1}$$

with

$$\eta_n^{(2)} = \xi_1\xi_{n+1} \quad (\mu = 0), \quad \eta_n^{(2)} = \xi_1 \quad (\mu = 1).$$

Here $\xi_1, \ldots, \xi_n, \tilde{\xi}_3, \ldots, \tilde{\xi}_{2\hat{m}-1}$ are mutually independent random variables with both ξ_k and $\tilde{\xi}_k$ being distributed as χ_k. Note that, except for the (asymptotically irrelevant) change in the factor $\eta_n^{(\beta)}$, the transition from the GOE to the GUE just amounts for splitting the terms ξ_k^2 into the products $\xi_k\tilde{\xi}_k$ of independent factors. It is precisely this split which causes the appearance of β in the denominator of the central limit theorem when written in the following form.

Theorem 6.1 (Tao and Vu [17, Theorem 4]). *With the notation as above there holds, as $n \to \infty$, the central limit theorem*

$$\frac{\log|\det M_n| - \frac{1}{2}\log n! + \frac{1}{4}\log n}{\sqrt{\frac{1}{\beta}\log n}} \xrightarrow{\mathrm{d}} N(0,1) \quad (\beta = 1, 2), \tag{6.2}$$

where $\xrightarrow{\mathrm{d}}$ *denotes convergence in distribution.*

To prove this theorem for $\beta = 1$ and $\beta = 2$ in parallel, we split

$$\log|\det M_n| = Y_n^{(\beta)} + Z_n^{(\beta)} \quad (\beta = 1, 2)$$

into the random variables $Y_n^{(\beta)} = \log \eta_n^{(\beta)}$ and $Z_n^{(\beta)}$ defined by

$$\begin{aligned} Z_n^{(1)} &= 2\log\xi_3 + 2\log\xi_5 + \cdots + 2\log\xi_{2\hat{m}-1}, \\ Z_n^{(2)} &= (\log\xi_3 + \log\tilde{\xi}_3) + (\log\xi_5 + \log\tilde{\xi}_5) + \cdots + (\log\xi_{2\hat{m}-1} + \log\tilde{\xi}_{2\hat{m}-1}), \end{aligned}$$

We immediately observe the relations

$$\mathrm{E}(Z_n^{(1)}) = \mathrm{E}(Z_n^{(2)}), \quad \mathrm{Var}(Z_n^{(1)}) = 2\mathrm{Var}(Z_n^{(2)}). \tag{6.3}$$

Note that the factor of two between the variances is caused, in the transition from GOE to GUE, by the above mentioned split of ξ_k^2 into the product $\xi_k\tilde{\xi}_k$.

Now, while proving the central limit theorem in the $\beta = 2$ case, Edelman and La Croix [7, Corollary 2] obtained, in passing, the following result.

Lemma 6.1. *The random variable $Z_n^{(\beta)}$ satisfies, as $n \to \infty$, a central limit theorem of the form*

$$\tilde{Z}_n^{(\beta)} = \frac{Z_n^{(\beta)} - \frac{1}{2}\log n! + \frac{1-\mu}{2}\log n + \frac{1}{4}\log n}{\sqrt{\frac{1}{\beta}\log n}} \xrightarrow{\mathrm{d}} N(0,1) \quad (\beta = 1, 2). \tag{6.4}$$

Proof. The proof of [7, Corollary 2] proceeds, first, by establishing asymptotic expansions based on explicit calculations of the mean and variance of $\log\chi$-distributed variables, namely,

$$\begin{aligned} \mathrm{E}(Z_n^{(2)}) &= \frac{1}{2}\log(2\hat{m}-1)! - \frac{1-\mu}{2}\log(2\hat{m}-1) - \frac{1}{4}\log(2\hat{m}-1) + O(1), \\ \mathrm{Var}(Z_n^{(2)}) &= \frac{1}{2}\log(2\hat{m}-1) + O(1), \end{aligned}$$

and, next, by showing that $Z_n^{(2)}$ satisfies a Lyapunov condition of order four. Hence, the Lindeberg–Feller central limit theorem can then be applied to $Z_n^{(2)}$ and gives, by noting that

$$\log(2\hat{m}-1)! = \log n! - (1-\mu)\log n, \quad \log(2\hat{m}-1) = \log n + O(1),$$

the asserted limit (6.4). By realizing that the sums $Z_n^{(1)}$ and $Z_n^{(2)}$ basically share the same Lyapunov condition, the central limit theorem for $Z_n^{(1)}$ can be induced from that of $Z_n^{(2)}$ by means of (6.3). □

The difference between the central limit theorems of $\log|\det M_n|$ and $Z_n^{(\beta)}$ enjoys the following strong convergence result.

Lemma 6.2. *The random variable $Y_n^{(\beta)}$ satisfies, as $n \to \infty$,*

$$\tilde{Y}_n^{(\beta)} = \frac{Y_n^{(\beta)} - \frac{1-\mu}{2}\log n}{\sqrt{\frac{1}{\beta}\log n}} \xrightarrow{\text{a.s.}} 0 \quad (\mu = 0, 1), \tag{6.5}$$

where $\xrightarrow{\text{a.s.}}$ denotes almost sure convergence.

Proof. The case $\mu = 1$ is trivial, since in that case $Y_n^{(\beta)}$ is independent of n. Applied to a sum of squares of independent standard Gaussians, the strong law of large numbers gives, as $n \to \infty$,

$$n^{-1}\xi_n^2 \xrightarrow{\text{a.s.}} \mathrm{E}\left(\xi_1^2\right) = 1, \quad \text{and, hence,} \quad n^{-1}(\xi_1^2 + 2\xi_n^2) \xrightarrow{\text{a.s.}} 2.$$

Taking the logarithm gives

$$\log \xi_n - \frac{1}{2}\log n \xrightarrow{\text{a.s.}} 0, \quad \log\sqrt{\xi_1^2 + 2\xi_n^2} - \frac{1}{2}\log n \xrightarrow{\text{a.s.}} \frac{1}{2}\log 2,$$

which implies the assertion for $\mu = 0$. □

Now, adding (6.4) and (6.5) gives, by Slutsky's theorem,

$$\frac{\log|\det M_n| - \frac{1}{2}\log n! + \frac{1}{4}\log n}{\sqrt{\frac{1}{\beta}\log n}} = \tilde{Y}_n^{(\beta)} + \tilde{Z}_n^{(\beta)} \xrightarrow{\text{d}} N(0,1) \quad (\beta = 1, 2),$$

which finishes the proof of the central limit theorem (6.2).

7. Integrating Out the Odd or Even Singular Values

Here, we present another proof of Theorem 1.1. If we are interested only in the distribution of the even singular values, then it is possible to proceed from the joint probability density (4.1) by integrating out the odd ones. While not exposing any additional structure, such as in Theorem 4.1, this approach is conceptually more straightforward, and offers the advantage that it can also be used to establish the determinantal formula (7.4) for the probability density of the odd singular values. This is of interest in its own right, since we constructed separate sparse random matrix models for the odd singular values in Corollary 4.2, Theorems 5.1 and 5.2. Moreover, the technique extends to the symmetric Jacobi and the Cauchy ensembles [3].

7.1. *Integrating out the odd singular values*

Recalling (1.5), we rewrite the expression (4.1) of the joint density in the form

$$q(s;t) = c_n 2^n n! \cdot g_\mu(s_1,\ldots,s_m) \cdot g_{1-\mu}(t_1,\ldots,t_{\hat m})$$

with functions

$$g_a(z_1,\ldots,z_m) = \prod_{k=1}^m z_k^a e^{-z_k^2/2} \cdot \Delta(z_m^2,\ldots,z_1^2). \tag{7.1}$$

Corollary 7.1 shows that integrating out the odd singular values t subject to the interlacing (2.4b) gives the following marginal density of the even singular values with $\delta_\mu = (\pi/2)^{\mu/2}$:

$$\begin{aligned} q_{\text{even}}(s_1,\ldots,s_m) &= \delta_\mu c_n 2^n n! \cdot g_\mu(s_1,\ldots,s_m)^2 \\ &= \delta_\mu c_n 2^n n! \cdot \prod_{k=1}^m s_k^{2\mu} e^{-s_k^2} \cdot \Delta(s_m^2,\ldots,s_1^2)^2. \end{aligned} \tag{7.2}$$

Since the last expression is the joint density of the anti-GUE of order n, see [15, Sec. 13.1] or [9, Exercise 1.3.5(iv)], this is nothing but Theorem 1.1 spelled out in terms of densities.

The integration is based on the following lemma and its first Corollary 7.1.

Lemma 7.1. *Let*

$$e_\kappa^{(n)}(x) = \begin{pmatrix} x^\kappa e^{-x^2/2} \\ x^{\kappa+2} e^{-x^2/2} \\ \vdots \\ x^{\kappa+2n-2} e^{-x^2/2} \end{pmatrix} \in \mathbb{R}^n \quad (\kappa = -1, 0, 1),$$

with the understanding that, instead of $x^{-1}e^{-x^2/2}$, the first entry of $e_{-1}^{(n)}(x)$ is the expression

$$\eta_{-1}(x) = -\sqrt{\frac{\pi}{2}}\,\mathrm{erf}\left(\frac{x}{\sqrt{2}}\right).$$

Then, for $\kappa = 0, 1$, there holds the integration formula

$$\int_{x_1}^{x_2} d\xi_1 \cdots \int_{x_n}^{x_{n+1}} d\xi_n \det\left(e_\kappa^{(n)}(\xi_1) \ \cdots \ e_\kappa^{(n)}(\xi_n)\right)$$

$$= \det\begin{pmatrix} e_{\kappa-1}^{(n)}(x_1) & \cdots & e_{\kappa-1}^{(n)}(x_{n+1}) \\ 1 & \cdots & 1 \end{pmatrix}.$$

Proof. Integration by parts yields the three-term recurrence of antiderivatives

$$\int^x e^{-\xi^2/2}\,d\xi = -\eta_{-1}(x),$$

$$\int^x \xi^{k+1} e^{-\xi^2/2}\,d\xi = -x^k e^{-x^2/2} + k\int^x \xi^{k-1} e^{-\xi^2/2}\,d\xi \quad (k=0,1,2,\ldots),$$

and, hence, by simplifying notation to $e_\kappa(x) = e_\kappa^{(n)}(x)$,

$$\int^x e_\kappa(\xi)\,d\xi = L_\kappa e_{\kappa-1}(x) \quad (\kappa = 0,1)$$

with a *lower triangular* matrix $L_\kappa \in \mathbb{R}^{n\times n}$ having -1 all along its main diagonal. We thus calculate

$$\begin{aligned}
&\int_{x_1}^{x_2} d\xi_1 \cdots \int_{x_n}^{x_{n+1}} d\xi_n \det(e_\kappa(\xi_1)\ \cdots\ e_\kappa(\xi_n))\\
&\quad = \det\left(\int_{x_1}^{x_2} e_\kappa(\xi_1)d\xi_1\ \cdots\ \int_{x_n}^{x_{n+1}} e_\kappa(\xi_n)d\xi_n\right)\\
&\quad = \underbrace{\det(L_\kappa)}_{=(-1)^n}\det(e_{\kappa-1}|_{x_1}^{x_2}\ \cdots\ e_{\kappa-1}|_{x_n}^{x_{n+1}})\\
&\quad = \det\begin{pmatrix} e_{\kappa-1}(x_1) & e_{\kappa-1}|_{x_1}^{x_2} & \cdots & e_{\kappa-1}|_{x_n}^{x_{n+1}}\\ 1 & 0 & \cdots & 0\end{pmatrix}\\
&\quad = \det\begin{pmatrix} e_{\kappa-1}(x_1) & e_{\kappa-1}(x_2) & \cdots & e_{\kappa-1}(x_{n+1})\\ 1 & 1 & \cdots & 1\end{pmatrix}.
\end{aligned}$$

In the last step we added the first column to the second, then the second to the third, etc. □

Corollary 7.1. *Let* g_μ *be as in* (7.1) *and put* $s_{\hat m} = 0$ *if* $\mu = 1$. *Then, one has the integration formula*

$$\int_{s_1}^{\infty} dt_1 \int_{s_2}^{s_1} dt_2 \cdots \int_{s_{\hat m}}^{s_{\hat m -1}} dt_{\hat m}\, g_{1-\mu}(t_1,\ldots,t_{\hat m}) = \delta_\mu\, g_\mu(s_1,\ldots,s_m) \quad (\mu = 0,1)$$

with $\delta_\mu = (\pi/2)^{\mu/2}$.

Proof. Using the notation of Lemma 7.1, we first observe that

$$g_\mu(z_1,\ldots,z_m) = \det(e_\mu^{(m)}(z_m)\ \cdots\ e_\mu^{(m)}(z_1)). \tag{7.3}$$

Now, Lemma 7.1 yields, first using $e_0^{(m)}(\infty) = 0$, that for $\mu = 0$

$$\int_{s_1}^{\infty} dt_1 \int_{s_2}^{s_1} dt_2 \cdots \int_{s_m}^{s_{m-1}} dt_m \det(e_1^{(m)}(t_m) \ \cdots \ e_1^{(m)}(t_1))$$

$$= \det\begin{pmatrix} e_0^{(m)}(s_m) & \cdots & e_0^{(m)}(s_1) & 0 \\ 1 & \cdots & 1 & 1 \end{pmatrix}$$

$$= \det(e_0^{(m)}(s_m) \ \cdots \ e_0^{(m)}(s_1))$$

and then, using $e_{-1}^{(m+1)}(0) = 0$ and $e_{-1}^{(m+1)}(\infty) = -(\sqrt{\pi/2}, 0)'$, that for $\mu = 1$

$$\int_{s_1}^{\infty} dt_1 \int_{s_2}^{s_1} dt_2 \cdots \int_{0}^{s_m} dt_{m+1} \det(e_0^{(m+1)}(t_{m+1}) \ \cdots \ e_0^{(m+1)}(t_1))$$

$$= \det\begin{pmatrix} 0 & e_{-1}^{(m+1)}(s_m) & \cdots & e_{-1}^{(m+1)}(s_1) & e_{-1}^{(m+1)}(\infty) \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$$

$$= (-1)^m \det(e_{-1}^{(m+1)}(s_m) \ \cdots \ e_{-1}^{(m+1)}(s_1) \quad e_{-1}^{(m+1)}(\infty))$$

$$= (-1)^m \det\begin{pmatrix} \eta_{-1}(s_m) & \cdots & \eta_{-1}(s_1) & -\sqrt{\frac{\pi}{2}} \\ e_1^{(m)}(s_m) & \cdots & e_1^{(m)}(s_1) & 0 \end{pmatrix}$$

$$= \sqrt{\frac{\pi}{2}} \det(e_1^{(m)}(s_m) \ \cdots \ e_1^{(m)}(s_1)),$$

which finishes the proof of the corollary. □

7.2. *Integrating out the even singular values*

The following second corollary of Lemma 7.1 will allow us to integrate out the *even* singular values from the density $q(s;t)$.

Corollary 7.2. *Let g_μ be as in* (7.1) *and put $t_{m+1} = 0$ if $\mu = 0$. Then, for $\mu = 0, 1$, one has the integration formula*

$$\int_{t_2}^{t_1} ds_1 \int_{t_3}^{t_2} ds_2 \cdots \int_{t_{m+1}}^{t_m} ds_m \, g_\mu(s_1, \ldots, s_m)$$

$$= \det\begin{pmatrix} e_{1-\mu}^{(\hat{m}-1)}(t_{\hat{m}}) & \cdots & e_{1-\mu}^{(\hat{m}-1)}(t_1) \\ \delta_{1-\mu}(t_{\hat{m}}) & \cdots & \delta_{1-\mu}(t_1) \end{pmatrix}$$

with $\delta_0(t) = 1$ and $\delta_1(t) = \sqrt{\pi/2}\,\mathrm{erf}(t/\sqrt{2})$.

Proof. Using (7.3) and Lemma 7.1 we obtain

$$\begin{aligned}
&\int_{t_2}^{t_1} ds_1 \cdots \int_{t_{m+1}}^{t_m} ds_m \, g_\mu(s_1, \ldots, s_m) \\
&\quad = \int_{t_2}^{t_1} ds_1 \cdots \int_{t_{m+1}}^{t_m} ds_m \, \det(e_\mu^{(m)}(s_m) \ \cdots \ e_\mu^{(m)}(s_1)) \\
&\quad = \det \begin{pmatrix} e_{\mu-1}^{(m)}(t_{m+1}) & \cdots & e_{\mu-1}^{(m)}(t_1) \\ 1 & \cdots & 1 \end{pmatrix},
\end{aligned}$$

which is already the assertion for $\mu = 1$. For $\mu = 0$, the assertion follows from further calculating

$$\begin{aligned}
\det &\begin{pmatrix} e_{-1}^{(m)}(t_{m+1}) & e_1^{(m)}(t_m) & \cdots & e_{-1}^{(m)}(t_1) \\ 1 & 1 & \cdots & 1 \end{pmatrix} \\
&= \det \begin{pmatrix} 0 & e_1^{(m)}(t_m) & \cdots & e_{-1}^{(m)}(t_1) \\ 1 & 1 & \cdots & 1 \end{pmatrix} \\
&= (-1)^m \det(e_{-1}^{(m)}(t_m) \ \cdots \ e_{-1}^{(m)}(t_1)) \\
&= (-1)^m \det \begin{pmatrix} -\delta_1(t_m) & \cdots & -\delta_1(t_1) \\ e_1^{(m-1)}(t_m) & \cdots & e_1^{(m-1)}(t_1) \end{pmatrix} \\
&= \det \begin{pmatrix} e_1^{(m-1)}(t_m) & \cdots & e_1^{(m-1)}(t_1) \\ \delta_1(t_m) & \cdots & \delta_1(t_1) \end{pmatrix}
\end{aligned}$$

which finishes the proof. □

Now, by means of this corollary, the marginal density of the odd singular values supported on $t_1 \geq t_2 \geq \cdots \geq t_{\hat{m}} \geq 0$ is given as

$$\begin{aligned}
q_{\mathrm{odd}}(t_1, \ldots, t_{\hat{m}}) &= c_n n! 2^n \cdot g_{1-\mu}(t_{\hat{m}}, \ldots, t_1) \cdot \det \begin{pmatrix} e_{1-\mu}^{(\hat{m}-1)}(t_{\hat{m}}) & \cdots & e_{1-\mu}^{(\hat{m}-1)}(t_1) \\ \delta_{1-\mu}(t_{\hat{m}}) & \cdots & \delta_{1-\mu}(t_1) \end{pmatrix} \\
&= c_n n! 2^n \cdot \det \begin{pmatrix} e_{1-\mu}^{(\hat{m}-1)}(t_{\hat{m}}) & \cdots & e_{1-\mu}^{(\hat{m}-1)}(t_1) \\ \gamma_{1-\mu}(t_{\hat{m}}) & \cdots & \gamma_{1-\mu}(t_1) \end{pmatrix} \\
&\quad \cdot \det \begin{pmatrix} e_{1-\mu}^{(\hat{m}-1)}(t_{\hat{m}}) & \cdots & e_{1-\mu}^{(\hat{m}-1)}(t_1) \\ \delta_{1-\mu}(t_{\hat{m}}) & \cdots & \delta_{1-\mu}(t_1) \end{pmatrix}
\end{aligned} \tag{7.4}$$

with

$$\gamma_\mu(t) = t^{\mu+2\hat{m}-2}e^{-t^2/2}, \quad \delta_\mu(t) = \begin{cases} 1 & \text{if } \mu = 0, \\ \sqrt{\frac{\pi}{2}}\,\mathrm{erf}\left(\frac{t}{\sqrt{2}}\right) & \text{if } \mu = 1. \end{cases}$$

Note that the two determinantal factors differ just in their last rows. It is this difference that prevents the expression from becoming a perfect square, which is in marked contrast with the marginal density of the even singular values.

8. Gap Probabilities

Theorem 1.1 has an interesting implication in terms of gap probabilities, that is, in terms of the probabilities

$$E^n_{\mathrm{RMT}}(k; J), \quad E^{\mathrm{limit}}_{\mathrm{RMT}}(k; J),$$

that the interval J contains exactly k eigenvalues drawn from the random matrix ensemble RMT of finite order n, or in some scaling limit. Here, RMT will be the GOE, the aGUE or the LUE with parameter a.

To begin with, by a simple change of coordinates, see [7, p. 8], there holds

$$E^{2m+\mu}_{\mathrm{aGUE}}(k; (0, s)) = E^m_{\mathrm{LUE}}(k; (0, s^2))\big|_{a=\mu-\frac{1}{2}} \quad (\mu = 0, 1). \tag{8.1}$$

By looking at pairs of consecutive values it is easy to see that the event that exactly k values of the decimated ensemble even$|\mathrm{GOE}_n|$, $n = 2m+\mu$, are contained in $(0, s)$ is given by the union of the events that exactly $2k+\mu-1$ or that exactly $2k+\mu$ values of $|\mathrm{GOE}_n|$ are in that interval. Since these two events are *mutually exclusive* and since the singular values of GOE contained in $(0, s)$ correspond to the eigenvalues in $(-s, s)$, we thus get from (1.2) and (8.1) proof of

$$\begin{aligned} &E^{2m+\mu}_{\mathrm{GOE}}(2k+\mu-1; (-s, s)) + E^{2m+\mu}_{\mathrm{GOE}}(2k+\mu; (-s, s)) \\ &\quad = E^{2m+\mu}_{\mathrm{aGUE}}(k; (0, s)) = E^m_{\mathrm{LUE}}(k; (0, s^2))\big|_{a=\mu-\frac{1}{2}} \quad (\mu = 0, 1). \end{aligned} \tag{8.2}$$

For even order ($\mu = 0$), a first proof of this formula was given by Forrester [8, Eq. (1.14)]. For odd order ($\mu = 1$), Forrester communicated to us further proof of the $k = 0$ case, a remarkable *tour de force* extending the techniques from [8] based on generating functions, Pfaffian calculus, and Fredholm determinants — later he was able to use the same approach to establish the general k case; for this and the extension to the symmetric Jacobi and the Cauchy ensembles, see [3].

We first identified the $\mu = 1$ form of (8.2) via a heuristic duality principle based on three observations. First, the LUE of order m and parameter $a = p - m \in \mathbb{N}$ is modeled by the eigenvalues of $m \times m$-Wishart matrices $W = X'X$, where the random $p \times m$-matrices X have independent *complex* standard normal entries. Second, the eigenvalues of $\tilde{W} = XX'$ are those of W padded with $a = p - m$ zeros; that is, the $(k+a)$th eigenvalue of $\tilde{W}$ is distributed as the kth eigenvalue of W. Last, since

$\tilde{W}$ is constructed the same way as W, but with dimension $\tilde{m} = m+a$ and parameter $\tilde{a} = -a$, we are thus led, at least *formally*, to the duality principle

$$E_{\mathrm{LUE}}^{m+\alpha}(k+\alpha;(0,t))|_{a=-\alpha} = E_{\mathrm{LUE}}^{m}(k;(0,t))|_{a=\alpha}. \tag{8.3}$$

Extrapolated to general $\alpha > -1$, it can be taken as a natural definition of an otherwise undefined expression. Now, formally evaluating the $\mu = 0$ form of (8.2) at half-integer values of m and k, and invoking the heuristic duality principle (8.3), led us to predict the $\mu = 1$ form. Since it held up under numerical scrutiny, trying to prove this prediction was a key motivation to our present work.

As already noted by Forrester [8, Eq. (1.16)], the bulk scaling of GOE and the hard-edge scaling of LUE allow us to turn (8.2), as $n \to \infty$, into the limit relation

$$\begin{aligned} &E_{\mathrm{GOE}}^{\mathrm{bulk}}(2k-1+\mu;(-s,s)) + E_{\mathrm{GOE}}^{\mathrm{bulk}}(2k+\mu;(-s,s)) \\ &\quad = E_{\mathrm{LUE}}^{\mathrm{hard}}\left(k;(0,\pi^2 s^2),\mu-\frac{1}{2}\right) \quad (\mu = 0,1); \end{aligned} \tag{8.4}$$

a remarkable formula previously established by Mehta [15, Eqs. (7.5.27), (7.5.29), (20.1.20) and (20.1.21)] using two different, but much more involved methods. In contrast to (8.4), which offers many advantages for the numerical calculation of gap probabilities of the GOE in the bulk scaling limit [2, Sec. 5], the finite-dimensional version (8.2) is not yet a closed recursion that would allow us to calculate the gap probabilities of the GOE on symmetric intervals: a complimentary expression evaluating

$$E_{\mathrm{GOE}}^{2m+\mu}(2k-\mu;(-s,s)) + E_{\mathrm{GOE}}^{2m+\mu}(2k+1-\mu;(-s,s)) \quad (\mu = 0,1)$$

is still missing. By the same arguments that justify (8.2) such an expression would establish the gap probabilities of the decimation ensemble $\mathrm{odd}|\mathrm{GOE}_n|$, whose joint distribution is given by (7.4).

To finish the paper, it is amusing to note that *all* three cases of the Tracy–Widom distributions

$$F_\beta(1;s) = E_\beta^{\mathrm{soft}}(0;(s,\infty)) \quad (\beta = 1,2,4),$$

can be sampled from the soft-edge scaling limit of the spectrum of just the GOE, i.e. the $\beta = 1$ case (the case $\beta = 2$ corresponds to the GUE, $\beta = 4$ to the GSE), see Fig. 1. First, let Λ_1, Λ_2 denote the largest and second-largest soft-edge scaled eigenvalues of the GOE. In the large-matrix limit, as $n \to \infty$, they are asymptotically distributed as

$$\Lambda_1 \overset{\mathrm{d}}{\sim} F_1(1;s), \quad \Lambda_2 \overset{\mathrm{d}}{\sim} F_4(1;s).$$

The first assertion is the definition of the distribution $F_1(1;s)$, while the second follows from the decimation relation, see [10, Theorem 5.2; 11, p. 44],

$$\mathrm{GSE}_m = \mathrm{even}\,\mathrm{GOE}_{2m+1}.$$

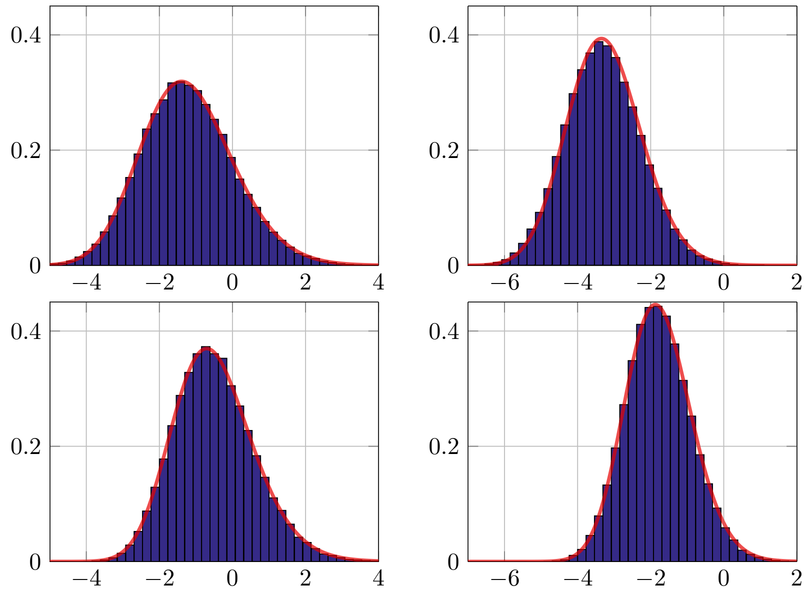

Fig. 1. Fluctuation statistics (densities) of 100,000 samples of GOE_n, $n = 50$, versus the theoretically predicted soft-edge scaling limits; top row: largest eigenvalue versus $F_1(1; s)$ (left), second-largest eigenvalue versus $F_4(1; s)$ (right); bottom row: largest singular value versus $F_1(1; s)^2$ (left); second-largest singular value versus $F_2(1; s)$ (right). We have consistently used the $O(n^{-2/3})$-scaling of Johnstone and Ma [14, Theorem 2].

Second, let Σ_1, Σ_2 denote the largest and second-largest scaled singular values of the GOE. They are asymptotically distributed as

$$\Sigma_1 \overset{\mathrm{d}}{\sim} F_1(1; s)^2, \quad \Sigma_2 \overset{\mathrm{d}}{\sim} F_2(1; s).$$

Here, the first assertion follows from the asymptotic independence of the extreme eigenvalues of the GOE and the second follows from (1.2) as follows: Σ_2 behaves like the largest scaled eigenvalue of the anti-GUE which, like that of the GUE, is governed by the Tracy–Widom distribution $F_2(1; s)$.

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References

[1] G. E. Andrews, I. P. Goulden and D. M. Jackson, Determinants of random matrices and Jack polynomials of rectangular shape, *Stud. Appl. Math.* **110**(4) (2003) 377–390.

[2] F. Bornemann, On the numerical evaluation of distributions in random matrix theory: A review, *Markov Process. Related Fields* **16**(4) (2010) 803–866.

[3] F. Bornemann and P. J. Forrester, Singular values and evenness symmetry in random matrix theory, preprint (2015), arXiv:1503.07383.

[4] R. Delannay and G. Le Caër, Distribution of the determinant of a random real-symmetric matrix from the Gaussian orthogonal ensemble, *Phys. Rev. E* (3) **62**(2, Part A) (2000) 1526–1536.

[5] L. Devroye, *Nonuniform Random Variate Generation* (Springer-Verlag, New York, 1986).

[6] I. Dumitriu and P. J. Forrester, Tridiagonal realization of the antisymmetric Gaussian β-ensemble, *J. Math. Phys.* **51**(9) (2010) 093302, 25 pp.

[7] A. Edelman and M. La Croix, The singular values of the GUE (less is more), preprint (2014), arXiv:1410.7065.

[8] P. J. Forrester, Evenness symmetry and inter-relationships between gap probabilities in random matrix theory, *Forum Math.* **18**(5) (2006) 711–743.

[9] P. J. Forrester, *Log-Gases and Random Matrices* (Princeton University Press, Princeton, NJ, 2010).

[10] P. J. Forrester and E. M. Rains, Interrelationships between orthogonal, unitary and symplectic matrix ensembles, in *Random Matrix Models and Their Applications* (Cambridge University Press, Cambridge, 2001), pp. 171–207.

[11] P. J. Forrester and E. M. Rains, Interpretations of some parameter dependent generalizations of classical matrix ensembles, *Probab. Theory Related Fields* **131**(1) (2005) 1–61.

[12] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 4th edn. (Johns Hopkins University Press, Baltimore, 2013).

[13] R. A. Horn and C. R. Johnson, *Matrix Analysis*, 2nd edn. (Cambridge University Press, Cambridge, 2013).

[14] I. M. Johnstone and Z. Ma, Fast approach to the Tracy–Widom law at the edge of GOE and GUE, *Ann. Appl. Probab.* **22**(5) (2012) 1962–1988.

[15] M. L. Mehta, *Random Matrices*, 3rd edn. (Elsevier/Academic Press, Amsterdam, 2004).

[16] S. Schechter, On the inversion of certain matrices, *Math. Tables Aids Comput.* **13** (1959) 73–77.

[17] T. Tao and V. Vu, A central limit theorem for the determinant of a Wigner matrix, *Adv. Math.* **231**(1) (2012) 74–101.

[18] R. C. Thompson, The behavior of eigenvalues and singular values under perturbations of restricted rank, *Linear Algebra Appl.* **13**(1/2) (1976) 69–78.