A topological method for rigorously computing periodic orbits using Fourier modes

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Abstract

We present a technique for the rigorous computation of periodic orbits in certain ordinary differential equations. The method combines set oriented numerical techniques for the computation of invariant sets in dynamical systems with topological index arguments. It not only allows for the proof of existence of periodic orbits but also for a precise (and rigorous) approximation of these. As an example we compute a periodic orbit for a differential equation introduced in [2].

Keywords. dynamical system, set oriented methods, periodic orbit, computer assisted proof

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1 Introduction

The problem of finding a periodic orbit of some ordinary differential equation can be reformulated in terms of finding a zero of a differential operator F on some suitably chosen space of periodic functions, see e.g. [4]. Using a particular basis $(\theta_k)_{k\in\mathbb{Z}}$ of this space one obtains a countable system of equations for the coefficients $(c_k)_{k\in\mathbb{Z}}$ of the periodic solution with respect to this basis. In order to prove the existence of a solution of this set of equations we essentially employ the following idea:

1. Numerically compute an approximate zero of F using a finite set of modes θ_k – this is the standard Galerkin approach.

2. Construct a restricted domain for F in the whole infinite dimensional space which isolates the numerical zero.

3. Using topological arguments, show that indeed the zero exists within the domain.

The domain acts as an outer approximation to the zero and provides an estimate on how good the approximation is. After having demonstrated the existence of the periodic orbit in Step 3 we can increase the precision of locating it by suitably shrinking the domain.

Our method has been inspired by recent work on rigorous computations for infinite dimensional dynamical systems: in [14] rigorous statements about the existence and location of equilibria in the Kuramoto-Sivashinsky equation are proved. In [3] methods are presented, that rigorously verify the existence of periodic orbits, connecting orbits and complicated dynamics in certain infinite dimensional discrete dynamical systems.

We are now going to indicate on how these three steps are carried out:

The first step of the method constitutes a well known numerical method for the approximate computation of a solution of some infinite dimensional system – the Galerkin projection approach. By projecting onto a finite dimensional subspace of the infinite dimensional space under consideration (the *finite modes*), one is effectively truncating the remaining ones (the *infinite modes*), i.e. one is setting them to zero. While it is possible to rigorously verify that a Galerkin projection of our operator has a zero, this is insufficient to conclude that F itself has a zero. This deficiency forces us to consider the truncated infinite modes – this is dealt with in the second step.

The second step consists in constructing a compact domain D for F which isolates the numerical zero from Step 1 – and eventually isolates a zero of F. The domain D being isolating means that there exists no zero of F on the boundary of D. The construction of D involves two parts: First we construct a domain for the finite modes by simply considering a (small) ball around the numerical zero. For the infinite modes we make use of a property of the Fourier coefficients of smooth solutions. Note that since the periodic orbit we are looking for inherits some smoothness properties from the underlying vector field, the magnitudes of its Fourier coefficients with respect to the basis (θ_k) have to decay at a certain rate with k. We can approximate this theoretic decay by considering the decay of the coefficients of the numerical zero. We choose a ball which is slightly bigger than a ball defined using this decay rate.

After having constructed D we have to verify that it is isolating. We first check for isolation in the infinite modes. Roughly speaking, if we have chosen a small enough domain D then on D the equation for each infinite mode is dominated by its linear part. More precisely we must estimate the non-linear part of the equation on D and show that it is small relative to the linear part (see Definition 1). In specific examples this *linear domination* can be exploited to reduce the question of isolation to one of isolation in the finite modes. We then treat the finite set of equations. By estimating the effect of the (truncated) infinite modes on the finite ones within the domain D we obtain bounds on the image of a point in the finite part of D. In this way we can verify if a particular point of the finite modes can or cannot be a zero of the full system. By doing so, we obtain a way of verifying if the finite modes isolate the zero. The verification of isolation in the finite modes is typically expensive from a computational point of view.

Note that there are several ways in which we can fail at obtaining isolation. We may have chosen too few finite modes, too big or small of a domain in the finite modes or we may have chosen too fast or slow of a decay rate in the infinite modes. In addition, if the reduction to the finite modes yields too many modes, then the computation may just take too long.

In the third and final step of the procedure we use topological arguments to guarantee the existence of a zero of F within D. This in fact requires only one additional fact beyond isolation: We can reconsider our zero finding problem as a fixed point problem by adding the identity to F. As long as the finite part of this new map has a non-trivial fixed point index on the finite part of D (see Appendix B), we can conclude that the entire map has a fixed point in D (and thus that F has a zero in D). The key reason for this is the treatment of the infinite modes in Step 2. In showing isolation in the infinite modes, we have in fact shown that a homotopy to a linear map (for each infinite mode) preserves the isolation of our domain. By piecing this together with the index of the Galerkin projection and homotopy invariance of the fixed point index we obtain the proof.

The paper is organized as follows: In Section 2 we give a general, yet more detailed and technical outline of the method in the particular context of a certain class of ordinary differential equations. This section also contains the main theorem. In Section 3 we explicitly perform the outlined computations using a specific example system. Section 4 contains the proof of the main theorem. In the appendix we lay down some technical estimates that we need for the computations in Section 3 and include some general background information on index theory, in particular Conley index theory.

2 General Outline

Our aim is to rigorously compute periodic solutions to an autonomous ordinary differential equation

$$F(y, y', y'', ..., y^{(r)}) = 0, \quad y(t) \in \mathbb{R}, \tag{1}$$

where $F : \mathbb{R}^{r+1} \to \mathbb{R}$ is some sufficiently smooth function. A typical procedure for the computation of periodic orbits is the geometric approach of determining fixed points of some suitably defined Poincaré map. Instead we here view the differential equation as an operator

$$F : y \mapsto F(y, y', y'', ..., y^{(r)}) \tag{2}$$

on a suitably defined space of periodic functions $y : \mathbb{R} \to \mathbb{R}$. Finding a zero of this operator is then equivalent to finding a periodic solution of the differential equation. In Section 3 we will apply this procedure to a specific equation, i.e. we will give a proof that the equation posesses a periodic orbit and determine several of its dominant Fourier modes accurately.

2.1 Constructing the Operator

The domain for (2) is the union over ω of the spaces $C^r_\omega = C^r_\omega(\mathbb{R}, \mathbb{R})$ of C^r-smooth real-valued periodic functions with frequency ω. In order to simplify this domain we rescale time within the differential operator by the variable frequency $\omega \in \mathbb{R}$. By doing so, we obtain a map

$$\begin{aligned} &\hat{F} : \mathbb{R} \times C^r_1 \to C^0_1, \\ &\hat{F} : (\omega, x) \mapsto F(x, \omega x', \omega^2 x'', ..., \omega^r x^{(r)}). \end{aligned}$$

As long as the map F is continuous, the operator $\hat{F}$ will be a continuous map from the Banach space $\mathbb{R} \times C^r_1$ to the Banach space C^0_1. The mapping $\hat{F}$ induces a map on the Fourier coefficients c_k, $k \in \mathbb{Z}$, of x. Note that since we deal with real-valued functions we have that $c_{-k} = \overline{c_k}$, $k \in \mathbb{Z}$, so we can restrict our attention to subsets of $\ell^2 = \ell^2(\mathbb{C}) = \left\{(c_k)_{k\geq 0} \in \mathbb{C}^{\mathbb{N}} : \sum_{k\geq 0} |c_k|^2 < \infty\right\}$. We treat ℓ^2 as the topological space defined by its standard norm. Let

$$\ell^2_r := \left\{(c_k)_{k\geq 0} \in \ell^2 \mid \exists u \in C^r_1 : \ c_k \text{ is the } k\text{-th Fourier coefficient of } u\right\}.$$

Clearly, the transform that assigns to each continuous function in C_1^r the corresponding ℓ_r^2 sequence is a bijection. We define the topology on ℓ_r^2 to be the quotient topology given by this bijection. Note that the inclusion map from ℓ_r^2 into ℓ^2 is a continuous injective mapping. Let $\tilde{F} : \mathbb{R} \times \ell_r^2 \to \ell_0^2$ denote the map induced from $\hat{F}$ on Fourier coefficients. If K is a compact set in ℓ_r^2 then $\tilde{F}$ restricted to $\mathbb{R} \times K$ is continuous from the subspace topology of K with respect to ℓ^2 into ℓ^2.

Since we are dealing with an autonomous differential equation, every T-periodic solution x of (1) gives rise to the continuum $\{x(\cdot + t) \mid t \in [0, T]\}$ of solutions. In order to cope with this numerically unfavorable feature we introduce a so called *phase condition* $\varphi : \ell_r^2 \to \mathbb{R}$ (see [6, 4]), such that $\varphi(c) = 0$ picks one particularly phased solution of $\tilde{F}(\omega, c) = 0$. The full system now reads

$$\begin{aligned} &\Phi : \mathbb{R} \times \ell_r^2 \to \mathbb{R} \times \ell_0^2, \\ &\Phi(\omega, c) = (\varphi(c), \tilde{F}(\omega, c)), \end{aligned}$$

and we are looking for solutions (ω, c) of

$$\Phi(\omega, c) = 0. \tag{3}$$

While it is now possible to treat this as a zero finding problem, we feel that there is more flexibility in treating it in a dynamical manner. We therefore introduce a mapping for which we are trying to find a fixed point. We then apply fixed point arguments in order to prove the existence of a zero of (3). To this end we define a map $G : \mathbb{R} \times \ell_r^2 \to \mathbb{R} \times \ell_0^2$ constructed from our map Φ such that

$$G(\omega, c) = (\omega, c) \quad \Leftrightarrow \quad \Phi(\omega, c) = 0.$$

For instance, we could consider

$$G = \iota + \Phi,$$

where ι is the inclusion map of $\mathbb{R} \times \ell_r^2$ into $\mathbb{R} \times \ell_0^2$. We will assume that G is continuous – this is satisfied if G is defined as above.

2.2 Finite Dimensional Reduction

For $k \geq 0$ define the projections

$$\begin{aligned} A_k : \mathbb{R} \times \ell^2 \to \mathbb{C}, \quad A_k(\omega, c) &= c_k, \\ P_k : \mathbb{R} \times \ell^2 \to \mathbb{R} \times \mathbb{C}^{k+1}, \quad P_k(\omega, c) &= (\omega, c_0, \ldots, c_k), \\ Q_k : \mathbb{R} \times \ell^2 \to \ell^2, \quad Q_k(\omega, c) &= (c_{k+1}, c_{k+2}, \ldots), \end{aligned}$$

as well as $A_\omega(\omega, c) = P_\omega(\omega, c) = \omega$.

By converting to a system on a Fourier space we have gone from a finite dimensional flow problem to an infinite dimensional map problem. The advantage and tractability of this approach in essences comes from Proposition 1 which reduces the problem to a countable sequence of finite dimensional problems. Let $\Omega \times D \subset \mathbb{R} \times \ell^2_r$ and define

$$\begin{aligned} Z_k(\Omega \times D) &= \left\{ (\omega, c) \in \Omega \times D : P_k G(\omega, c) = P_k(\omega, c) \right\}, \\ Z(\Omega \times D) &= \bigcap_{k \geq 0} Z_k(\Omega \times D). \end{aligned}$$

Proposition 1. *Suppose that G is continuous, $Z_k(\Omega \times D) \neq \emptyset$ for all $k \geq 0$ and $Z_M(\Omega \times D)$ is compact for some M. Then $Z(\Omega \times D) \neq \emptyset$ and for all $(\omega, c) \in Z(\Omega \times D)$, $G(\omega, c) = (\omega, c)$.*

Proof. In order to show that $Z(\Omega \times D)$ is non-empty, we only need to show that the $Z_k(\Omega \times D)$ form a nested sequence of compact sets. The nesting property, i.e., $Z_{k+1}(\Omega \times D) \subset Z_k(\Omega \times D)$, is clear from the definition. The set $Z_k(\Omega \times D)$ is defined as the pre-image of a closed set under a continuous map and therefore closed. If $k \geq M$, it is compact by the fact that it is a closed set contained in a compact set. Therefore $Z(\Omega \times D)$ is non-empty.

Fix $(\omega, c) \in Z(\Omega \times D)$. If (ω, c) does not satisfy $G(\omega, c) = (\omega, c)$ then there exists an index i for which $A_i G(\omega, c) \neq A_i(\omega, c)$. But $(\omega, c) \in Z_k(\Omega \times D)$ for all k and in particular for $k = i$. This is a contradiction and therefore $G(\omega, c) = (\omega, c)$. □

So far we recasted our original problem of finding a periodic orbit for an ordinary differential equation into the problem of finding a fixed point for some (infinite-dimensional) map. Via Proposition 1 we reduced this question to a sequence of finite dimensional problems. As suggested by Proposition 1 the proof proceeds in two steps:

The first step in computing a fixed point of G is estimating its location by using Newton's method applied to a Galerkin projection of G. If the vector field of the underlying differential equation is of class C^r, we know that all solutions will also be of class C^r. This in turn means that their Fourier coefficients c_k decay at a rate $\mathcal{O}(k^{-r})$ as $k \to \infty$. From Proposition 1 recall that we need the set Z_M to be compact. In order to guarantee that Z_M is compact, we restrict the domain of G to a *compact* subset $\Omega \times D$ of $\mathbb{R} \times \ell_r^2$. We will choose a compact set

$$D \subset \prod_{k=0}^{\infty} \{c_k \in \mathbb{C} : |c_k| \leq r_k\}$$

in such a way that it contains our estimate for the fixed point for $k \leq M$ and also r_k decays at a rate consistent with the smoothness of the solutions of the differential equation.

2.3 Proving the Existence of the Periodic Solution

We are now going to use tools from algebraic topology in order to show that the sets Z_k are nonempty. As suggested by Proposition 1 we will first show that $Z_M \neq \emptyset$ for some (small) M. Then we lift the information to the sets Z_k for $k > M$.

Define the finite dimensional multivalued map

$$\mathcal{G}_M : P_M(\Omega \times D) \rightrightarrows P_M(\mathbb{R} \times \ell^2), \tag{4}$$

$$(\omega, \bar{c}) \mapsto \{P_M \circ G(\omega, c) : P_M(\omega, c) = (\omega, \bar{c}) \text{ and } (\omega, c) \in \Omega \times D\}.$$

Using the Conley index theory we are going to prove that every continuous selector (of some suitable enclosure) of this map has a fixed point (see the Appendix for a definition of these notions). Note that in particular the map

$$(\omega, c_0, \ldots, c_M) \mapsto P_M \circ G(\omega, c_0, \ldots, c_M, \bar{c}_{M+1}, \bar{c}_{M+2}, \ldots)$$

is a continuous selector of $\mathcal{G}_M$ for every fixed $(\bar{c}_{M+1}, \bar{c}_{M+2} \ldots) \in Q_M(\Omega \times D)$ – meaning that Z_M is nonempty.

Note that by construction of $\mathcal{G}_M$ the set $Q_M(\Omega \times D)$ determines the size of the images of $\mathcal{G}_M$. In order to be able to efficiently analyze this map, we need to make their images as small as possible. Therefore a proper choice of

$\Omega \times D$ will be one which contains the fixed point but also makes the images of $\mathcal{G}_M$ as small as possible.

In order to apply the Conley index theory to our fixed point problem we have to find a domain which forms an isolating neighborhood for the higher order coefficients. To this end we need the following notion:

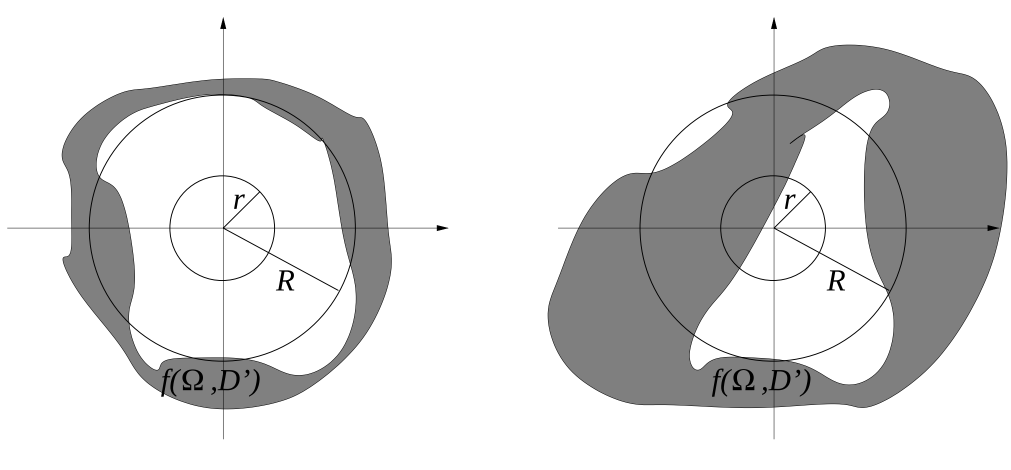
Definition 1. A map $f : \Omega \times D \to \mathbb{C}$ is *linearly dominated* on the ball $B_0(r) \subset A_k(\Omega \times D)$ if

$$f(\omega, c) = L(\omega)c_k + g(\omega, c),$$

for all $(\omega, c) \in \Omega \times D$, $|c_k| \leq r$, where $L : I \to \mathbb{C}$, $\Omega \subset I$, and g are continuous functions, such that

$$\sup_{\Omega \times D} |g(\omega, c)| < r \inf_{\omega \in I} |L(\omega)| - r. \tag{5}$$

The idea of the definition is quite simple: The map f is linearly dominated if it has a small non-linear part relative to the linear part which expands uniformly over Ω.



(a) Linearly dominated. (b) Not linearly dominated.

Figure 1: Linear domination. $R = r \inf_{\omega \in I} |L(\omega)|$, $D' = D \cap \{|c_k| = r\}$.

In practice, by choosing a good domain $\Omega \times D$ we can show that the maps $A_k G$ are linearly dominated for large k. This completes the proof of the existence of a periodic orbit as stated by the following theorem, which is going to be proved in Section 4.

Theorem 1. *Let $\Omega \times D$ be compact. Suppose that there exists a star-shaped enclosure $\hat{\mathcal{G}}_M$ for $\mathcal{G}_M$ such that (N, L) is an index pair for $\hat{\mathcal{G}}_M$ and*

$$\Lambda((N, L), \hat{\mathcal{G}}_M) \neq 0,$$

where Λ is the Lefschetz number of the index pair (N, L). Assume furthermore that the maps $A_k G$ are linearly dominated for all $k > M$, then there exists a fixed point for the map G.

3 An Example Computation

In this section we show that the differential equation

$$y''' + y'' + \sigma y' - \delta y + y^2 = 0 \tag{6}$$

with parameters $\sigma = 2$ and $\delta = 3$ has a (nontrivial) periodic orbit. This example will clarify and illustrate the procedure laid out in the previous section.

3.1 Constructing the Operator and its Domain

The induced map $\tilde{F}$ on Fourier coefficients takes the form

$$A_k \tilde{F} : (\omega, c) \mapsto (-i\omega^3 k^3 - \omega^2 k^2 + \sigma\omega i k - \delta) c_k + \sum_{l \in \mathbb{Z}} c_l c_{k-l}, \quad k \geq 0. \tag{7}$$

For the later construction of G we are going to use a slightly rewritten version of $\tilde{F}$, where we work with real coordinates $c_k = a_k + ib_k$ for $k \leq 2$. To this end define $A_k^r(\omega, c) = \text{real}(c_k) = a_k$ and $A_k^i(\omega, c) = \text{imag}(c_k) = b_k$. The map $\tilde{F}$ then reads

$$A_k^r \tilde{F} : (\omega, c) \mapsto (-\omega^2 k^2 - \delta) a_k + (\omega^3 k^3 - \sigma\omega k) b_k + \sum_{l \in \mathbb{Z}} (a_l a_{k-l} - b_l b_{k-l}), \tag{8}$$

$$A_k^i \tilde{F} : (\omega, c) \mapsto (-\omega^3 k^3 + \sigma\omega k) a_k + (-\omega^2 k^2 - \delta) b_k + \sum_{l \in \mathbb{Z}} (a_l b_{k-l} + b_l a_{k-l}),$$

$k \geq 0$. We chose $\varphi(c) = \text{imag}(c_1) = b_1 = 0$ as the phase condition – this is a standard choice (see [4]) and turns out to be numerically favorable. We use Φ as defined in the previous section.

Before we proceed to the construction of G let us choose its domain $\Omega \times D$ first. By applying Newton's method to a Galerkin projection of Φ we obtain a guess for ω and the first nine modes of the solution:

$$
\begin{array}{lcrlcll}
\omega & = & & 1.39, & & & \\
c_0 & = & & 2.46, & & & \\
c_1 & = & & 0.813, & & & \\
c_2 & = & & 0.0130 & - & i & 0.0361, \\
c_3 & = & - & 7.79 \cdot 10^{-4} & - & i & 5.13 \cdot 10^{-4}, \\
c_4 & = & - & 1.31 \cdot 10^{-5} & + & i & 1.23 \cdot 10^{-5}, \\
c_5 & = & & 1.55 \cdot 10^{-7} & + & i & 2.64 \cdot 10^{-7}, \\
c_6 & = & & 4.55 \cdot 10^{-9} & - & i & 1.46 \cdot 10^{-9}, \\
c_7 & = & - & 6.51 \cdot 10^{-12} & - & i & 7.04 \cdot 10^{-11}, \\
c_8 & = & - & 9.95 \cdot 10^{-13} & - & i & 1.25 \cdot 10^{-13}, \\
c_9 & = & - & 4.65 \cdot 10^{-15} & + & i & 1.29 \cdot 10^{-14}.
\end{array}
$$

Figure 2 shows the decay of $|c_k|$ with k (dots). We use this approximate

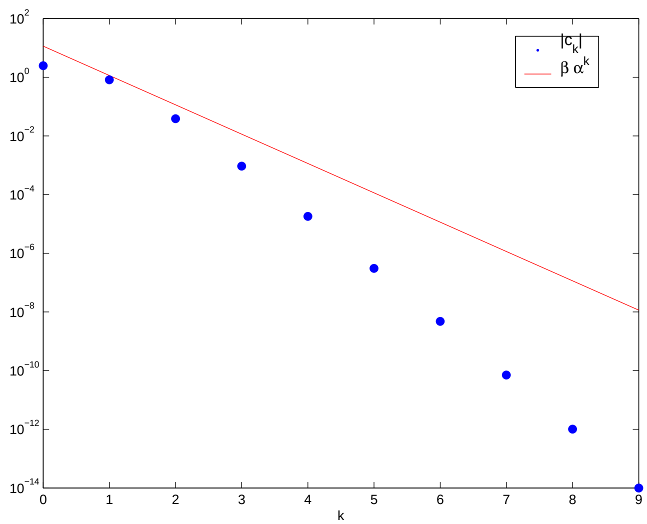


Figure 2: Magnitude of the Fourier modes of a guessed solution.

solution to construct a domain with an exponential decay rate:

$$\begin{aligned}
\Omega &= [1.36, 1.44], \\
D &= D_0 \times D_1 \times D_2 \times \prod_{k=3}^{\infty} B_0(\beta\alpha^k), \\
\text{with } \alpha &= 0.1, \quad \beta = 11.5, \\
\text{and } D_0 &= [2.43, 2.49], \\
D_1 &= [0.8, 0.8285], \\
D_2 &= [-0.003, 0.04] - [0.005, 0.06]i,
\end{aligned}$$

see Figure 2. The domain D is compact in C_1^3 and contains the supposed zero. Note that $\text{imag}(D_0) = 0$ since the Fourier series is real valued and $\text{imag}(D_1) = 0$ due to the phase condition. The differential equation (6) has two trivial periodic solutions, namely the two fixed points of the system. We have taken care to avoid these fixed points in the chosen domain. We now define G as in the previous section by $G : \Omega \times D \to \mathbb{R} \times \ell_\infty^2$, $G = \iota + \Phi$.

3.2 The Existence Proof

We begin by showing that the maps A_kG are linearly dominated for $k > M = 2$. As shown in Appendix A we have the estimate

$$\left| \sum_{l \in \mathbb{Z}} c_l c_{k-l} \right| \leq \beta^2 \alpha^k \left[\frac{2}{1-\alpha^2} + k - 1 \right]. \tag{9}$$

for the nonlinear part of G on the chosen domain. This estimate enables us to prove the following result.

Lemma 2. *The map*

$$A_kG$$

is linearly dominated on the ball $B_0(\beta\alpha^k)$ for $k > 2$.

Proof. By applying the estimate (9) to the map A_kG we obtain that for $k > 0$

$$A_kG(\omega, c) \in c_k + (-i\omega^3k^3 - \omega^2k^2 + \sigma\omega ik - \delta)c_k + B_0\left(\beta^2\alpha^k \left[\frac{2}{1-\alpha^2} + k - 1\right]\right)$$

In order to show that the map $A_k G$ is linearly dominated on $B_0(\beta\alpha^k)$, we need to verify condition (5), which explicitely reads in our example:

$$\beta^2\alpha^k\left[\frac{2}{1-\alpha^2}+k-1\right] < \beta\alpha^k \min_{\omega\in\Omega}\left(|-i\omega^3k^3-\omega^2k^2+\sigma\omega ik-\delta+1|-1\right). \tag{10}$$

We can rewrite the right hand side of the inequality above:

$$\begin{aligned}&\min_{\omega\in\Omega}(|-i\omega^3k^3-\omega^2k^2+\sigma\omega ik-\delta+1|-1)\\&\quad=\min_{\omega\in\Omega}(\sqrt{\omega^4k^4+2\omega^2k^2(\delta-1)+(\delta-1)^2+k^6\omega^6-2k^4\omega^4\sigma+k^2\sigma^2\omega^2}-1)\end{aligned}$$

We will write $\Omega=[\underline{\omega},\overline{\omega}]$. By replacing ω by $\underline{\omega}$ in each positive term in the square root and by $\overline{\omega}$ in each negative term we obtain a lower bound for the minimum.

$$\begin{aligned}&\sqrt{\underline{\omega}^4k^4+2\underline{\omega}^2k^2(\delta-1)+(\delta-1)^2+k^6\underline{\omega}^6-2k^4\overline{\omega}^4\sigma+k^2\sigma^2\underline{\omega}^2}-1\\&\leq\min_{\omega\in\Omega}(\sqrt{\omega^4k^4+2\omega^2k^2(\delta-1)+(\delta-1)^2+k^6\omega^6-2k^4\omega^4\sigma+k^2\sigma^2\omega^2}-1)\end{aligned}$$

For $k\geq 3$,

$$\begin{aligned}&\sqrt{\underline{\omega}^4k^4+2\underline{\omega}^2k^2(\delta-1)+(\delta-1)^2+k^6\underline{\omega}^6-2k^4\overline{\omega}^4\sigma+k^2\sigma^2\underline{\omega}^2}-1\\&\qquad-\beta\left[\frac{2}{1-\alpha^2}+k-1\right]\end{aligned}$$

is an increasing function in k and for $k=3$, it is greater than zero. Therefore, inequality (10) holds for $k>2$. □

We now turn to the proof that every continuous selector of some suitable enclosure $\hat{\mathcal{G}}_M$ of the multivalued map (4) has a fixed point for $M=2$. As already mentioned we will do this by using the Conley index theory, i.e. we are going to construct an index pair for $\hat{\mathcal{G}}_2$ which has a non-zero Lefschetz number.

First note that if $\Phi(\omega,c)=0$, $(\omega,c)\in\Omega\times D$, then

1. $b_0=0$, since $c_0=\overline{c_0}$;
2. $b_1=0$ due to the phase condition;

3. we can solve $A_0^r \tilde{F}(\omega, c) = 0$ for a_0 within our domain,

$$a_0 = a_0(Q_0 c) = \frac{\delta}{2} + \sqrt{\frac{\delta^2}{4} - \sum_{l \in \mathbb{Z},\, l \neq 0} (a_l a_{-l} - b_l b_{-l})},$$

   and we get that $a_0(Q_0 c) \in \mathrm{real}(D_0)$ for $c \in D$;

4. and *simulatenously* we can solve $A_1^r \tilde{F}(\omega, c) = 0$ for ω,

$$\omega = \omega(Q_0 c) = \sqrt{-\delta + \frac{1}{a_1} \sum_{l \in \mathbb{Z}} (a_l a_{1-l} - b_l b_{1-l})},$$

   again we get $\omega(Q_0 c) \in \Omega$ for $c \in D$.

So by restricting the map G to the compact subset

$$\Gamma = \{(\omega, c) \in \Omega \times D \mid \omega = \omega(Q_0 c), a_0 = a_0(Q_0 c), b_0 = 0, b_1 = 0\}$$

we have effectively reduced the dimension of the problem by four, i.e.

$$P_2(\Gamma) \cong \mathbb{R}^3,$$

so that for $M = 2$ we are left with only a three dimensional system. Note that we are working with a real version of the map G, see (8) for the explicit formulae. The (restricted) multivalued enclosure $\hat{\mathcal{G}}_2$ on $P_2(\Gamma)$ that we use in the computations is as follows:

$$\begin{aligned}
a_1 &\mapsto (-\bar{\omega}^2 - \delta + 1)a_1 + 2a_1 a_2 + 2\bar{a}_0 a_1 + |c_2|\ I_0 + 0.01\ I_1 \\
a_2 &\mapsto (-4\bar{\omega}^2 - \delta + 1)a_2 + (8\bar{\omega}^3 - 2\sigma\bar{\omega})b_2 + 2\bar{a}_0 a_2 + a_1^2 \\
&\qquad + (|c_2|\ 0.1 + |c_1|)I_0 + 10^{-3}\ I_1 \\
b_2 &\mapsto (-8\bar{\omega}^3 + 2\sigma\bar{\omega})a_2 + (-4\bar{\omega}^2 - \delta + 1)b_2 + 2\bar{a}_0 b_2 \\
&\qquad + |c_2| 0.1\ I_0 + |c_1| I_0 + 10^{-3}\ I_1,
\end{aligned}$$

where

$$I_0 = 0.023 \cdot [-1, 1], \quad I_1 = 2.67172 \cdot 10^{-4} \cdot [-1, 1],$$

and $\bar{a}_0 = \bar{a}_0(a_1, a_2, b_2)$ and $\bar{\omega} = \bar{\omega}(a_1, a_2, b_2)$ are intervals. These intervals and the enclosure $\hat{\mathcal{G}}_2$ have been computed based on the estimates in Appendix A, in particular using inequality (11). Note that the image of a given point under $\hat{\mathcal{G}}_2$ is the product of intervals.

In order to construct an index pair (N, L) for $\hat{\mathcal{G}}_2$ we are going to use the methods described in [13]. For a brief description of the corresponding notions we refer to Appendix B. These methods use a discretization of (part of the) phase space into a *cubical grid* $\mathcal{P}$, i.e. a partition into closed cubes ("boxes"). The two compact sets $N, L \subset \mathbb{R}^d$ will be represented as subsets of this grid, i.e. they will be constituted by the union of the corresponding cubes.

We first compute an isolating neighborhood within the cube

$$\begin{aligned} D' &:= \mathrm{real}(D_1) \times \mathrm{real}(D_2) \times \mathrm{imag}(D_2) \\ &= [0.8, 0.8285] \times [-0.003, 0.04] \times -[0.005, 0.06] \subset \mathbb{R}^3 \end{aligned}$$

by computing a (cubical) covering of the maximal invariant set in D'. In doing so we use a partition $\mathcal{P}$ of D' consisting of 4096 boxes and a multivalued map $\mathbf{G}_M : \mathcal{P} \rightrightarrows \mathcal{P}$ such that for $C \in \mathcal{P}$ we have $\hat{\mathcal{G}}_2(C) \subset |\mathbf{G}_M(C)|$, where for $\mathcal{C} \subset \mathcal{P}$ we define

$$|\mathcal{C}| = \bigcup_{C \in \mathcal{C}} C.$$

$\mathbf{G}_M$ assigns a set of cubes to each cube in the grid as its image. We compute the maximal invariant set $\mathcal{I}$ of $\mathbf{G}_M$ as a set of cubes and, by checking the criterion given in [13], show that $|\mathcal{I}|$ is isolating for $\hat{\mathcal{G}}_2$. Figure 3 shows the result of the computation. Following [13] a corresponding index pair for $\hat{\mathcal{G}}_M$ is then given by

$$(N, L) := (|\mathbf{G}_M(\mathcal{I})|, |\mathbf{G}_M(\mathcal{I}) \backslash \mathcal{I}|).$$

We then compute the homology map corresponding to the map induced by $\hat{\mathcal{G}}_M$ on (N, L) (see Appendix B). This can be done using only the grid map $\mathbf{G}_M$, see [1, 3] for details. The induced homology map is the identity on level three homology and zero otherwise. Therefore the Lefschetz number for the index pair is $\Lambda((N, L), \hat{\mathcal{G}}_2) = 1$. This implies that $|\mathcal{I}|$ contains a fixed point for every continuous selector of $\mathcal{G}_2$.

Combining this with Lemma 2 we have shown the existence of a fixed point $(\bar{\omega}, \bar{c})$ of G in the domain $\Omega \times D$. By construction of G the point $(\bar{\omega}, \bar{c})$ satisfies

$$\Phi(\bar{\omega}, \bar{c}) = 0$$

thus proving the existence of a periodic orbit of the differential equation (6).

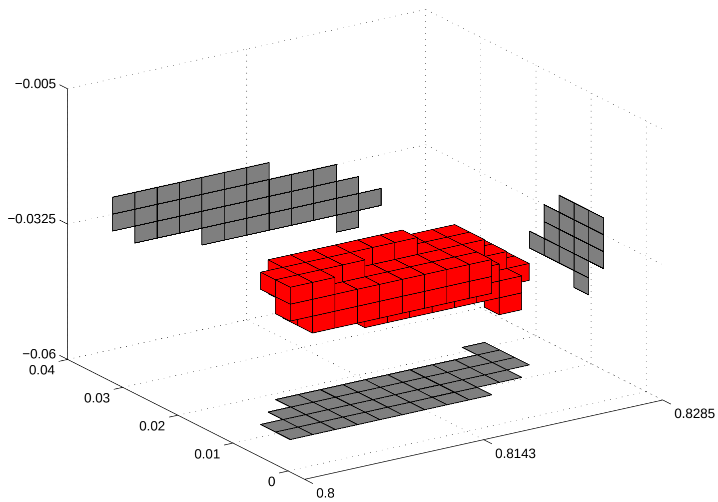


Figure 3: Isolating neighborhood for $\hat{\mathcal{G}}_2$.

Theorem 3. *The differential equation (6) with parameters $\sigma = 2$ and $\delta = 3$ possesses a periodic orbit*

$$y(t) = \sum_{k \in \mathbb{Z}} c_k e^{ik\omega t}$$

with $c_{-k} = \overline{c_k}$ and

$$\begin{aligned}
\omega &\in [1.36, 1.44] \\
c_0 &\in [2.43, 2.49] \\
c_1 &\in [0.8, 0.8285] \\
c_2 &\in [-0.003, 0.04] - [0.005, 0.06]i \\
c_k &\in \{z \in \mathbb{C} \mid |z| \leq \beta \cdot \alpha^k\}, \quad \beta = 11.5, \quad \alpha = 0.1, \quad k > 2.
\end{aligned}$$

Remark 1. Once a periodic orbit has been located, one can in principle compute its Fourier coefficients up to arbitrary precision. To this end one applies a subdivison algorithm [5] to the collection $\mathcal{I}$, where in the selection step one computes the maximal invariant set of $\mathbf{G}_M$ on the refined collection. Simultaneously one can also tighten the bounds on the "higher order modes" $c_k, k > 2$, see [14, 3] for more details.

4 A Fixed Point Theorem

In this section we prove Theorem 1 as a corollary of a slightly more general fixed point theorem. We begin by considering the general setting for that theorem.

4.1 Linear Domination Revisited

Let B be a (real) Banach space such that

$$B = \prod_{k \in \mathcal{I}} B_k.$$

over a finite or countable index set $\mathcal{I}$ such that $d_k := \dim(B_k) < \infty$ for all k. Let D be a compact subset of B such that

$$D = \prod_{k \in \mathcal{I}} D_k$$

and $D_k \subset B_k$ is a neighborhood retract (and therefore a Euclidean neighborhood retract). Let $G : D \to B$ be a continuous map. Let A_I, $I \subset \mathcal{I}$, be the natural projection from B onto $\prod_{k \in I} B_k$ and $A_k = A_{\{k\}}$. Again, following Section 2, we define the multivalued map $\mathcal{G}_I$ as

$$\mathcal{G}_I : \bar{c} \mapsto \{A_I \circ G(c) : A_I(c) = \bar{c}, c \in D\}.$$

We redefine the notion of linear domination in this more general setting – still consistent with the earlier definition. Let $g : D \to D$, $p : D \to K$ and $L : K \to \mathcal{L}(B_k)$ be continuous functions, where K is a contractible topological space and $\mathcal{L}(B_k)$ is the space if linear operators on B_k. Further restrict L such that for every $\kappa \in K$ the linear map $L(\kappa) : B_k \to B_k$ takes the form $L(\kappa) = TQ$, where $Q \in SO(d_k)$ is a rotation and T is a dilation, i.e. T corresponds to a diagonal matrix $\mathrm{diag}(e_1, \ldots, e_{d_k})$. Define $\|L(\kappa)\|_{\min} = \min_{i=1,\ldots,d_k} |e_i|$.

Definition 2. The map $A_k G : D \to B_k$ is *linearly dominated* on the ball $B_0(r) \subset D_k$ if

$$A_k G(c) = L(p(c)) A_k(c) + g(c),$$

for all $c \in D$ such that $A_k(c) \in B_0(r)$, and

$$\sup_{c \in D} |g(c)| < r \inf_{\kappa \in K} \|L(\kappa)\|_{\min} - r.$$

Remark 2. There is a natural homotopy for linearly dominated maps to a simpler linearly dominated map. Let $\mathcal{C} : [0,1] \times K \to K$ be the contraction of K to the point $\bar{k}$. We define the *linearly dominated homotopy* $\mathcal{LH}$ of the map $A_k G$ to be the map $\mathcal{LH} : [0,1] \times D \to B_k$ such that

$$(c,t) \mapsto L(\mathcal{C}(t, p(c))) A_k(c) + (1-t) g(c).$$

This is a homotopy from $A_k G$ to $L(\bar{k}) A_k(c)$. Note that for each $t \in [0,1]$ the map $\mathcal{LH}(t, \cdot)$ is linearly dominated on the same ball $B_0(r)$.

The goal of the remainder of this section is to prove the following main theorem which has Theorem 1 as a corollary.

Theorem 4. *Let I be a finite subset of $\mathcal{I}$. Let (N_I, L_I) be an index pair for a star-shaped enclosure $\hat{\mathcal{G}}_I$ of the multivalued map $\mathcal{G}_I$. If $\Lambda(\hat{\mathcal{G}}_I, (N_I, L_I)) \neq 0$ and for all $k \notin I$, $A_k G$ is linearly dominated then there exists a fixed point for the map G within the set*

$$(N_I \backslash L_I) \times \prod_{k \in \mathcal{I} \backslash I} D_k.$$

4.2 Induced Multivalued Maps and Product Spaces

We show – in a multivalued context – how to compute the fixed point index of a map $f : U_1 \times U_2 \to X_1 \times X_2$ that is "almost the product of two maps". We can write $f := (f_{U_1}, f_{U_2})$, where $f_{U_i} : U_1 \times U_2 \to X_i$ for $i = 1, 2$.

Definition 3. A multivalued map F_{U_i} is induced by $f : U_1 \times U_2 \to X_1 \times X_2$ if the following diagram commutes in the multivalued sense (i.e. replacing $=$ with $\in$ where appropriate).

$$\begin{array}{ccc} U_1 \times U_2 & \xrightarrow{f} & X_1 \times X_2 \\ \Big\downarrow {\scriptstyle \pi_{U_i}} & & \Big\downarrow {\scriptstyle \pi_{X_i}} \\ U_i & \xrightarrow{\overset{F_{U_i}}{\rightrightarrows}} & X_i. \end{array}$$

Observe that if F_{U_1} is induced by f then F_{U_1} is independent of f_{U_2}. Note also that f is a continuous selector of $F_{U_1} \times F_{U_2}$.

Using the standard product of sets we define the product of the pairs $U = (U_1, U_0)$ and $V = (V_1, V_0)$ by

$$U \times V := (U_1 \times V_1, U_0 \times V_1 \cup U_1 \times V_0).$$

Lemma 5. *Let U and V be topological spaces and let $f : U \times V \to X \times Y$ be continuous. Let F_U and F_V be the multivalued maps induced by f. Let (N_U, L_U) and (N_V, L_V) be index pairs for F_U and F_V. Then $(N, L) := (N_U, L_U) \times (N_V, L_V)$ is an index pair for f.*

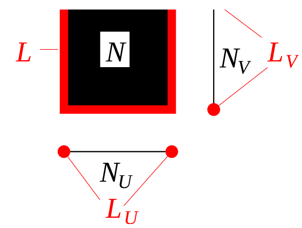


Figure 4: Product index pair

Proof. Observe that

$$N_U \times N_V \setminus (L_U \times N_V \cup N_U \times L_V) \;=\; (N_U \setminus L_U) \times (N_V \setminus L_V)\,.$$

This equality is sufficient to show part (1) and (3) of the definition of index pairs (Definition 4). Part (2) follows from the following inclusion:

$$\begin{aligned}
f(L_U \times N_V \;\cup\; & L_V \times N_U) \cap (N_U \times N_V) \\
\subset\; & (F_U \times F_V(L_U \times N_V) \cup F_U \times F_V(N_U \times L_V)) \cap (N_U \times N_V) \\
=\; & ((F_U(L_U) \times F_V(N_V)) \cup (F_U(N_U) \times F_V(L_V))) \cap (N_U \times N_V) \\
=\; & ((F_U(L_U) \cap N_U) \times (F_V(N_V) \cap N_V)) \\
& \cup ((F_U(N_U) \cap N_U) \times (F_V(L_V) \cap N_V)) \\
\subset\; & (L_U \times N_V) \cup (N_U \times L_V)
\end{aligned}$$

□

Lemma 6. *With the conditions from Lemma 5, if U and V are contractible then $f_{N/L}$ is homotopic to a product map $(g_U \times g_V)_{N/L}$.*

Proof. Let $\mathcal{H}_U$ and $\mathcal{H}_V$ be the respective contraction homotopies of U and V to the points $\overline{u} \in U$ and $\overline{v} \in V$. Define the maps $g_U : U \to X$ by $g_U(u) := f_U(u, \overline{v})$ and $g_V : V \to Y$ by $g_V(v) := f_V(\overline{u}, v)$.

Consider the homotopy $\mathcal{H} : [0,1] \times U \times V \to X \times Y$ such that

$$(t, u, v) \mapsto (f_U(u, \mathcal{H}_V(t, v)), f_V(\mathcal{H}_U(t, u), v)).$$

From the definition of F_U it is clear that $f_U(u, \mathcal{H}_V(t, v)) \in F_U(u)$ for all $t \in [0,1]$. Similarly, the same holds for the V component. From Lemma 11 and Remark 4, we get the desired result. □

Lemma 7. *With the conditions from Lemma 5, if F_U and F_V are star-shaped multivalued maps then $f_{N/L}$ is homotopic to a product map $(g_U \times g_V)_{N/L}$.*

Proof. The proof follows from the fact that any two continuous selectors of a star-shaped multivalued map are homotopic. Using the same terminology as in the proof above f and $g_U \times g_V$ are both continuous selectors of the map $F_U \times F_V$. □

Lemma 8. *Suppose that F_U is a star-shaped multivalued map induced by $f : U \times V \to X \times Y$ and that f_V is linearly dominated on $B_0(r) \subset V$. Let (N_U, L_U) be an index pair for F_U. Then $(N, L) := (N_U, L_U) \times (B_0(r), \partial B_0(r))$ is an index pair for f and $f_{N/L}$ is homotopic to a product map $(g_U \times g_V)_{N/L}$, where g_V is a linear expanding map.*

Proof. Since f_V is linearly dominated on $B_0(r) \subset V$, $(B_0(r), \partial B_0(r))$ is an index pair for F_V. Consider the linearly dominated homotopy $\mathcal{LH}$ of f_V. For every $t \in [0,1]$, the pair $(B_0(r), \partial B_0(r))$ is an index pair for $\mathcal{LH}(t, \cdot)$. In addition consider the homotopy $\mathcal{H}_U : [0,1] \times U \times V \to X$ of f_U to g_U (the U component of $\mathcal{H}$ defined in the proof of Lemma 6). For every $t \in [0,1]$ the map $\mathcal{H}_U(t, \cdot)$ is a continuous selector of F_U, therefore (N_U, L_U) is an index pair for $\mathcal{H}_U(t, \cdot)$ for every $t \in [0,1]$. Similarly the combination of the two homotopies preserves the index pair (N, L). From Lemma 11 and Remark 4, we get the desired result. □

Corollary 1. *Suppose that F_U is a star-shaped multivalued map induced by*

$$f : U \times \prod_{i=1}^{n} V_i \to X \times \prod_{i=1}^{n} Y_i$$

and that f_{V_i} is linearly dominated on $B_0(r_i) \subset V_i$ for all $i = 1, 2, \ldots, n$. Let (N_U, L_U) be an index pair for F_U. Then

$$(N, L) := (N_U, L_U) \times \prod_{i=1}^{n} (B_0(r_i), \partial B_0(r_i))$$

is an index pair for f and $f_{N/L}$ is homotopic to a product map $(g_U \times \prod_{i=1}^{n} g_{V_i})_{N/L}$, where g_{V_i} is a linear expanding map.

The proof is given by an obvious extending in notation of Lemma 8.

Remark 3. Let us examine g_V of Lemma 8 more closely. Note again that $(B_0(r), \partial B_0(r))$ is an index pair for g_V, a map represented by a diagonal matrix with all diagonal entries greater than 1 in magnitude. From this fact we can conclude that $|\Lambda(g_V, (B_0(r), \partial B_0(r))| = 1$. From Theorem 12 this implies that $\mathrm{Ind}(g_V, \mathrm{int}\, B_0(r)). \neq 0$

4.3 Proof of Theorem 4

By an adaptation of Proposition 1, it is sufficient to prove the theorem for a finite index set $\mathcal{I}$. We can rewrite the theorem in the notation of Corollary 1, i.e. $f = G$, $U = \prod_{k \in I} D_k$, etc. – adding that $\Lambda(F_U, (N_U, L_U)) \neq 0$. We invoke Theorem 12 several times to compute $\mathrm{Ind}(f, \mathrm{int}(N))$ – first, $\mathrm{Ind}(f, \mathrm{int}(N)) = \Lambda(f, (N, L))$. By the homotopy invariance of the Lefschetz number,

$$\Lambda(f, (N, L)) = \Lambda(g_U \times \prod_{i=1}^{n} g_{V_i}, (N, L)).$$

Again, by Theorem 12,

$$\Lambda(g_U \times \prod_{i=1}^{n} g_{V_i}, (N, L)) = \mathrm{Ind}(g_U \times \prod_{i=1}^{n} g_{V_i}, \mathrm{int}(N)).$$

By the multiplicity properties of the fixed point index,

$$\Lambda(f, (N, L)) = \mathrm{Ind}(g_U, \mathrm{int}(N_U)) \times \prod_{i=1}^{n} \mathrm{Ind}(g_{V_i}, \mathrm{int}\, B_0(r)).$$

From Remark 3, the product $\prod_{i=1}^{n} \mathrm{Ind}(g_{V_i}, \mathrm{int}\, B_0(r)) \neq 0$ and once again by Theorem 12 $\mathrm{Ind}(g_U, \mathrm{int}(N_U)) \neq 0$. Therefore, by the Wazewski's property for the fixed point index, the map f (i.e. G) has a fixed point — q.e.d.

A Estimates

In this appendix we prove the estimates necassary for the construction of the multivalued map (4) and for the proof of Lemma 2. For general ordinary differential equations, such estimates will form the backbone of its analysis.

Lemma 9. *Let $(c_k)_{k \in \mathbb{Z}}$ be a sequence of complex numbers such that $c_{-k} = \overline{c_k}$ and*

$$|c_i| \leq \beta \alpha^i$$

for $\alpha < 1$, $\beta > 0$ and for all $i \geq M > 0$. Then for $0 \leq \kappa \leq 2M$

$$\left| \sum_{\substack{|l|>M \\ or\ |\kappa-l|>M}} c_l c_{\kappa-l} \right| \leq 2\beta \sum_{-M \leq l < \kappa - M} |c_l| \alpha^{\kappa-l} + \beta^2 \alpha^\kappa \left[\frac{2\alpha^{2M+2}}{1-\alpha^2} \right] \tag{11}$$

Proof. We regroup the sum:

$$\begin{aligned}
\sum_{\substack{|l|>M \\ \text{or } |\kappa-l|>M}} c_l c_{\kappa-l} &= \sum_{\substack{|l|\leq M \\ |\kappa-l|>M}} c_l c_{\kappa-l} + \sum_{\substack{|l|>M \\ |\kappa-l|\leq M}} c_l c_{\kappa-l} + \sum_{\substack{|l|>M \\ |\kappa-l|>M}} c_l c_{\kappa-l} \\
&= 2 \sum_{\substack{|l|\leq M \\ |\kappa-l|>M}} c_l c_{\kappa-l} + \sum_{\substack{|l|>M \\ |\kappa-l|>M}} c_l c_{\kappa-l} \\
&= 2 \sum_{-M\leq l<\kappa-M} c_l c_{\kappa-l} + \sum_{l<-M} c_l c_{\kappa-l} + \sum_{l>\kappa+M} c_l c_{\kappa-l} \\
&= 2 \sum_{-M\leq l<\kappa-M} c_l c_{\kappa-l} + 2 \sum_{l>M} c_{-l} c_{\kappa+l}
\end{aligned}$$

From the triangle inequality and using the given inequalities for all c_k, we get:

$$\begin{aligned}
\left| \sum_{\substack{|l|>M \\ \text{or } |\kappa-l|>M}} c_l c_{\kappa-l} \right| &\leq 2\beta \sum_{-M\leq l<\kappa-M} |c_l| \alpha^{|\kappa-l|} + 2\beta^2 \sum_{l>M} \alpha^{|l|+|\kappa+l|} \\
&= 2\beta \sum_{-M\leq l<\kappa-M} |c_l| \alpha^{\kappa-l} + 2\beta^2 \alpha^{\kappa} \frac{\alpha^{2M+2}}{1-\alpha^2}
\end{aligned}$$

□

Lemma 10. *Let $(c_k)_{k\in\mathbb{Z}}$ be a sequence of complex numbers such that $c_{-k} = \overline{c_k}$ and*

$$|c_i| \leq \beta\alpha^i$$

for $\alpha < 1$, $\beta > 0$ and for all $i \geq 0$. If $\kappa \geq 0$ then

$$\left| \sum_{l\in\mathbb{Z}} c_l c_{\kappa-l} \right| \leq \beta^2 \alpha^{\kappa} \left[\frac{2}{1-\alpha^2} + (\kappa - 1) \right]. \tag{12}$$

Proof.

$$\begin{aligned}
\sum_{l\in\mathbb{Z}} c_l c_{\kappa-l} &= \sum_{l\leq 0} c_l c_{\kappa-l} + \sum_{0<l<\kappa} c_l c_{\kappa-l} + \sum_{l\geq\kappa} c_l c_{\kappa-l} \\
&= \sum_{0<l<\kappa} c_l c_{\kappa-l} + 2 \sum_{l\geq 0} c_{-l} c_{\kappa+l}
\end{aligned}$$

From the triangle inequality and substituting the inequality above where possible we get:

$$\begin{aligned}\left|\sum_{l\in\mathbb{Z}} c_l c_{\kappa-l}\right| &\leq \beta^2 \sum_{0<l<\kappa} \alpha^{|l|+|\kappa-l|} + 2\beta^2 \sum_{l\geq 0} \alpha^{|l|+|\kappa+l|} \\ &= \beta^2 \sum_{0<l<\kappa} \alpha^{\kappa} + 2\beta^2 \sum_{l\geq 0} \alpha^{2l+\kappa} \\ &= \beta^2\alpha^{\kappa}(\kappa-1) + 2\beta^2\alpha^{\kappa}\frac{1}{1-\alpha^2}\end{aligned}$$

□

B Index Theory

In this section we give a brief introduction to index theory and multivalued maps.

A *topological pair* (X, A) is an ordered pair where X is a topological space and A is a subspace of X. The map $f : (X, A) \to (Y, B)$ is a *map on topological pairs* if $f(A) \subset B$. A *pointed space* (X, x_0) is an ordered pair where X is a topological space and x_0 is a point of X. A *pointed map* $f : (X, x_0) \to (Y, y_0)$ is a continuous map such that $f(x_0) = y_0$.

There is a natural correspondence between topological pairs and pointed spaces. To observe this we require the concept of a quotient space. Given a pair (X, A) we define the space X/A to be the pointed space $(X \setminus A \cup [A], [A])$ where $[A]$ is a distinguished point outside of the set X. The point $[A]$ should be thought of as the collapsing of the set A to a single point. The topology on X/A is defined by the projection map $\pi_A : X \to X/A$

$$\pi_A(x) = \begin{cases} x & x \in X \setminus A, \\ [\mathrm{A}] & \text{otherwise} \end{cases}$$

In particular, a set U is open in X/A if and only if $\pi^{-1}(U)$ is open in X. Clearly the topology on X/A is the quotient topology induced by π_A if the set A is non-empty. Note that there is an excisive property to the quotient space X/A, i.e. if A is closed and $U \subset A$ is open then X/A is homeomorphic to $(X \setminus U)/(A \setminus U)$.

If $f : (X, A) \to (Y, B)$ is a map on pairs then the *induced map* $\tilde{f} : X/A \to Y/B$ is defined as

$$\tilde{f}(x) = \begin{cases} \pi_B(f(x)) & x \in X/A \setminus [A], \\ [\mathrm{B}] & \text{otherwise.} \end{cases}$$

It is easy to check that the pointed map $\tilde{f}$ is continuous. In the case that X/A is homeomorphic to Y/B via some map h, the mapping $h \circ \tilde{f}$ can be viewed as a time-discrete dynamical system – this system will be denoted by $f_{X/A}$.

A map on topological pairs induces a map on homology of pairs which is denoted

$$f_* : H_*(X, A) \to H_*(Y, B).$$

In addition $\tilde{f}$ induces a map on homology

$$\tilde{f}_* : H_*(X/A) \to H_*(Y/B).$$

The maps f_* and $\tilde{f}_*$ are isomorphic in the case that A and B are neighborhood deformation retracts within their respected spaces.

B.1 Multivalued Maps

A natural way of doing rigorous computations on real numbers on a computer is by dealing with outer approximation of the objects under consideration. Often computer assisted proofs use interval arithmetic as a way of outer approximating a point (or set). In the context of (time-discrete) dynamical systems one faces the problem of rigorously computing the image of a given point under – typically – a continuous map. The idea of using an outer approximation of the image point naturally leads to the notion of a multivalued map. In this section we define a specific class of multivalued maps and show how they are useful in the study of a dynamical system.

A *multivalued map* $F : X \rightrightarrows Y$ is a map from a toplogical space X to the power set $\mathcal{P}(Y)$, Y a toplogical space. For our purposes, we define the graph of a multivalued map $F : X \rightrightarrows Y$ to be

$$\Gamma_F = \{(x, y) \in X \times Y : y \in F(x)\}.$$

A map $G : X \rightrightarrows Y$ is a *selector* of F if $\Gamma_G \subset \Gamma_F$, F is referred to as an *enclosure* for G. In the case that G is single-valued and continuous it is a *continuous selector*.

A set $A \subset X$ is *acyclic*, if it has reduced homology zero – for example if A is contractible (to a point) within X. A multivalued map F is *acyclic* if the image of any point is acyclic and F has at least one continuous selector. The importance of this definition is that an acyclic multivalued map induces a single valued homomorphism on homology, i.e. given a multivalued map $F : X \rightrightarrows Y$ then $F_* : H_*(X) \to H_*(Y)$. In addition, any continuous selector f of F has an isomorphic homological map, i.e.

$$F_* \approx f_*.$$

A multivalued map $F : X \rightrightarrows Y$, Y a subset of a vector space, is called *star-shaped* if there exists a continuous selector f such that for any continuous selector g and for all $x \in X$ the line segment joining $f(x)$ and $g(x)$ is contained in $F(x)$. As with acyclic multivalued maps, star-shaped multivalued maps have the property that they induce a homomorphism on homology. Indeed, in many ways this is a stronger condition then being acyclic. In particular any two continuous selectors of a star-shaped multivalued map are in fact homotopic to one another. The homological map of a star-shaped multivalued map is defined to be the homological map of any continuous selector of F.

We note that the product $F_1 \times F_2$ of two acyclic (resp. star-shaped) maps is again acyclic (resp. star-shaped).

In addition star-shaped multivalued maps work well on topological pairs, i.e. pairs (X_1, X_0), $X_0 \subset X_1$, X_1 a topological space. For a multivalued map $F : X_1 \rightrightarrows Y_1$ we write $F : (X_1, X_0) \rightrightarrows (Y_1, Y_0)$, if $F(X_i) \subset Y_i$ for $i = 1, 2$ and call F a map on pairs. Note that if F is map on pairs, then every continuous selector of F is again a map on (the same) pairs. It is clear that for any two continuous selectors of F there is a homotopy between them which preserves the topological pair property. We can therefore define the homological map of F to be the homology of any continuous selector.

B.2 Index Pairs for the Conley Index

The use of multivalued maps in itself can tell us very little about the dynamics of our system and in particular the existence of fixed points. In order to obtain useful information we will compute homological invariants of subsets of the domain of our multivalued map. The Conley index is the tool that is most natural for such a study. The index pair is the primary definition of

Conley index theory which we will focus on. For a thorough treatment of the Conley index see [10, 11, 9].

For a multivalued map $F : U \subset X \rightrightarrows X$ we define an orbit σ_x to be a sequence $\sigma_x = (x_k)_{k\in\mathbb{Z}}$, $x_k \in U$, such that $x_k \in F(x_{k-1})$ and $x_0 = x$. A set $S \subset U$ is *invariant* if for all $x \in S$ there exists an orbit $\sigma_x = (x_k)_{k\in\mathbb{Z}}$ such that $x_k \in S$ for all k. The *maximal invariant set* within a set $N \subset U$, denoted by $\mathrm{Inv}_F(N)$, is the set of all points $x \in N$ such that an orbit σ_x stays in N. Note that if f is a continuous selector of F, then $\mathrm{Inv}_f(N) \subset \mathrm{Inv}_F(N)$.

An *isolating neighborhood* is a compact set $N \subset U$ such that its maximal invariant set lies in its interior; i.e.,

$$\mathrm{Inv}_F(N) \subset \mathrm{int}(N).$$

A set S is an *isolated invariant set* if $S = \mathrm{Inv}_F(N)$ for some isolating neighborhood N.

Definition 4. A pair of compact sets (N, L) where $L \subset N \subset U$ is an *index pair* for F if

1. $S = \mathrm{Inv}_F(\mathrm{cl}(N \setminus L)) \subset \mathrm{int}(N \setminus L)$,
2. $F(L) \cap N \subset L$,
3. $F(N \setminus L) \cap N \setminus L \subset \mathrm{int}(N \setminus L)$.

If we consider a single-valued selector f of the map F with index pair (N, L), then

$$f : (N, L) \to (N \cup F(N), (N \cup F(N)) \setminus (N \setminus L))$$

is a topological map on pairs. By the excisive properties of quotient spaces and the properties of an index pair, N/L is homeomorphic to $(N \cup F(N))/((N \cup F(N)) \setminus (N \setminus L)))$ and so $f_{N/L}$ is a continuous time-discrete dynamical system. We refer to $f_{N/L}$ as the *index map* for the index pair (N, L) and the map f.

We can extend the definition of index map to a star-shaped multivalued map F by considering the homotopy type of the map $f_{N/L}$. This is due to the fact that all other continuous selectors g of F have index maps which are homotpic.

Existence of an index pair for a stronger definition of isolating invariant set is shown in [10]. We do not use this definition since we are not concerned with the question of existence for every isolated invariant set. Instead, we will compute index pairs and then infer the existence of an isolated invariant set.

The most powerful aspect of Conley index theory is its continuation properties i.e. the index does not change under homotopies which preserve the isolation of a neighborhood. We state a simple version of the continuation property in the following lemma.

A homotopy $\mathcal{H} : [0,1] \times U \to X$ is said to *preserve the index pair* (N, L) if (N, L) is an index pair for $\mathcal{H}(\tau, \cdot)$ for all fixed $\tau \in [0, 1]$.

Lemma 11. *Let $f, g : U \to X$. Suppose that f and g are homotopic via the map $\mathcal{H}$ which preserves the index pair. Then $f_{N/L}$ is homotopic to $g_{N/L}$.*

Remark 4. We apply this lemma many times in Section 4 in the following manner. Consider F a multivalued map with index pair (N, L) such that f and g are both continuous selectors of F and are homotopic by the map h (as in Lemma 11). If $\mathcal{H}([0,1], x) \subset F(x)$ then (N, L) is an index pair for $\mathcal{H}(\tau, \cdot)$ for all fixed $\tau \in [0, 1]$. So, $f_{N/L}$ is homotopic to $g_{N/L}$.

We are interested in the existence of fixed points for our system. We will briefly discuss the fixed point index defined in [8] and some of its properties. We will then relate how to use the index map to determine the fixed point index and thereby prove the existence of fixed points by properties of the index map.

To begin we define the Lefschetz number of f. Assume that $\varphi := \{\varphi_i\}$ is an endomorphism of a graded vector space over the field of rational or real numbers. The *generalized rank* of an endomorphism $\varrho : E \to E$, E a vector space, is the dimension of $E' := E/(\cup \{\varrho^{-n}(0) : n = 1, 2, 3, ...\})$. If ϱ has finite generalized rank then the trace of ϱ, $tr(\varrho)$ is defined to be $tr(\varrho')$ where $\varrho' : E' \to E'$ is the induced map. The map φ is of *finite type* if and only if the generalized rank of φ_i is finite for all i. If φ is of finite type the *Lefschetz number* of φ is

$$\Lambda(\varphi) := \sum_{n=0}^{\infty} (-1)^n tr \varphi_n$$

If (N, L) is an index pair for f, then we call (N, L) of finite type if f_{N/L_*} is of finite type and define $\Lambda(f, (N, L)) := \Lambda(f_{N/L}) := \Lambda(f_{N/L_*})$.

Let $f : U \subset X \to X$ be a map such that X is an absolute neighborhood retract. Define $\mathrm{Fix}_f(V)$ to be the set of fixed points of f within the set

$V \subset U$. The map $f|_V$ is said to be *admissible* if V is open and $\mathrm{Fix}_f(V)$ is compact. For admissible maps, it is possible to define a fixed point index (see [7],[8]), Ind, such that $\mathrm{Ind}(f, V) := \mathrm{Ind}(f|_V)$ is an integer, uniquely characterized by seven properties. We list three properties here that such an index posesses:

1. (Wazewski's property) If $\mathrm{Ind}(f, V) \neq 0$ then $\mathrm{Fix}_f(V) \neq \emptyset$.
2. (Multiplicity) Let $f_1 : U_1 \to X_1$, $f_2 : U_2 \to X_2$ and $f_1 \times f_2 : U_1 \times U_2 \to X_1 \times X_2$. If $f_1|_{V_1}$ and $f_2|_{V_2}$ are admissible then so is $f_1|_{V_1} \times f_2|_{V_2}$ and
$$\mathrm{Ind}(f_1 \times f_2, V_1 \times V_2) = \mathrm{Ind}(f_1, V_1) \cdot \mathrm{Ind}(f_2, V_1).$$
3. (Continuation) Let $\mathcal{H} : [0, 1] \times U \to X$ be a homotopy from $\mathcal{H}_0$ to $\mathcal{H}_1$ such that $\mathrm{Fix}_{\mathcal{H}}(V) := \cup_{0 \leq t \leq 1} \mathrm{Fix}_{\mathcal{H}_t}(V)$ is compact ($\mathcal{H}_t := \mathcal{H}(t, \cdot)$). Then
$$\mathrm{Ind}(\mathcal{H}_0, V) = \mathrm{Ind}(\mathcal{H}_1, V).$$

The following theorem relates the fixed point index to the Lefschetz number of an index map.

Theorem 12 (Szymczak [12]). *Let $U \subset X$ be a Euclidean neighborhood retract and $S \subset U$ an isolated invariant set with respect to a continuous map $f : U \to X$. Then there exists an index pair (N, L) of finite type and*
$$\mathrm{Ind}(f, \mathrm{int}(N)) = \Lambda(f, (N, L)).$$

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