# Solutions to Worksheet 3

**Exercise 1:**
For the first exercise recall that the Taylor-Expansion of $f(x+h)$ around $f(x)$ is given by

$$f(x+h) = \sum_{i=0}^{N} \frac{h^i}{i!} f^{(i)}(x) + \mathcal{O}(h^{N+1}). \tag{0.1}$$

- In the beginning we fix the second component and write down the Taylor expansion of $f(x+h)$ and $f(x-h)$ up to third order:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x) + \mathcal{O}(h^4)$$
$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x) + \mathcal{O}(h^4)$$

  So, if we sum up both expressions, terms with odd power in $h$ will cancel out.

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + \mathcal{O}(h^4)$$

  Solving the equation to $f''(x)$ leads to the given approximation.

$$h^2 f''(x) = f(x+h) + f(x-h) - 2f(x) + \mathcal{O}(h^4)$$
$$f''(x) = \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} + \mathcal{O}(h^2)$$

  The same holds true for partial derivatives, i.e.

$$\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{f(x+h,y) + f(x-h,y) - 2f(x,y)}{h^2} + \mathcal{O}(h^2).$$

  For the Laplacian, this means

$$\Delta f = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right)(x,y) \tag{0.2}$$
$$= \frac{f(x+h,y) + f(x-h,y) - 2f(x,y)}{h^2} + \frac{f(x,y+h) + f(x,y-h) - 2f(x,y)}{h^2} + \mathcal{O}(h^2) \tag{0.3}$$
$$= \frac{1}{h^2}(f(x+h,y) + f(x,y+h) + f(x-h,y) + f(x,y-h) - 4f(x,y)) + \mathcal{O}(h^2) \tag{0.4}$$

- To deduce an "Order 4"-method, we proceed as in the previous case. We start again with Taylor expansion.

$$f(x+h) = f(x) + hf^{(1)}(x) + \frac{h^2}{2} f^{(2)}(x) + \frac{h^3}{6} f^{(3)}(x) + \frac{h^4}{24} f^{(4)}(x) + \frac{h^5}{120} f^{(5)}(x) + \mathcal{O}(h^6)$$
$$f(x-h) = f(x) - hf^{(1)}(x) + \frac{h^2}{2} f^{(2)}(x) - \frac{h^3}{6} f^{(3)}(x) + \frac{h^4}{24} f^{(4)}(x) - \frac{h^5}{120} f^{(5)}(x) + \mathcal{O}(h^6)$$
$$f(x+2h) = f(x) + 2hf^{(1)}(x) + 2h^2 f^{(2)}(x) + \frac{4}{3} h^3 f^{(3)}(x) + \frac{2}{3} h^4 f^{(4)}(x) + \frac{4}{15} h^5 f^{(5)}(x) + \mathcal{O}(h^6)$$
$$f(x-2h) = f(x) - 2hf^{(1)}(x) + 2h^2 f^{(2)}(x) - \frac{4}{3} h^3 f^{(3)}(x) + \frac{2}{3} h^4 f^{(4)}(x) - \frac{4}{15} h^5 f^{(5)}(x) + \mathcal{O}(h^6)$$

We sum again up the expressions for "$\pm h$" to cancel out all terms with odd order in $h$,

$$f(x+h)+f(x-h) = 2f(x)+h^2 f^{(2)}(x)+\frac{1}{12}h^4 f^{(4)}(x)+\mathcal{O}(h^6),$$
$$f(x+2h)+f(x-2h) = 2f(x)+4h^2 f^{(2)}(x)+\frac{4}{3}h^4 f^{(4)}(x)+\mathcal{O}(h^6).$$

The last step is to add up both sums such that the fourth derivative vanishes.

$$16(f(x+h)+f(x-h))-(f(x+2h)+f(x-2h)) = 30f(x)+12h^2 f^{(2)}(x)+\mathcal{O}(h^6)$$
$$12h^2 f^{(2)}(x) = 16(f(x+h)+f(x-h))-(f(x+2h)+f(x-2h))-30f(x)+\mathcal{O}(h^6)$$
$$f^{(2)}(x) = \frac{1}{h^2}(\frac{4}{3}(f(x+h)+f(x-h))-\frac{1}{12}(f(x+2h)+f(x-2h))-\frac{5}{2}f(x))+\mathcal{O}(h^4)$$

Using this results for both partial derivatives $\frac{\partial^2}{\partial x^2}f$ and $\frac{\partial^2}{\partial y^2}f$ the nine-point stencil of the Laplacian reads

$$\begin{aligned}\Delta f =&(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2})(x,y)\\ \approx&\frac{1}{h^2}\left(\frac{4}{3}(f(x+h,y)+f(x,y+h)+f(x-h,y)+f(x,y-y))\right.\\ &\left.-\frac{1}{12}(f(x+2h,y)+f(x,y+2h)+f(x-2h,y)+f(x,y-2h))-5f(x,y)\right).\end{aligned}$$

- We are given the function $u(x,y) = e^{\pi x}\sin(\pi y)+0.5(xy)^2$. Inserting the given value shows that $u$ satisfies on the boundary

$$\begin{aligned}u(0,y) &= \sin(\pi y),\\ u(x,0) &= 0,\\ u(1,y) &= e^{\pi}\sin(\pi y)+0.5y^2,\\ u(x,1) &= 0.5x^2.\end{aligned}$$

Moreover, the derivatives of $u$ are

$$\begin{aligned}u_x(x,y) &= \pi e^{\pi x}\sin(\pi y)+xy^2,\\ u_y(x,y) &= \pi e^{\pi x}\cos(\pi y)+x^2 y,\\ u_{xx}(x,y) &= \pi^2 e^{\pi x}\sin(\pi y)+y^2,\\ u_{yy}(x,y) &= \pi^2 e^{\pi x}(-\sin(\pi y))+x^2.\end{aligned}$$

So the Laplacian satisfies

$$\Delta u = (u_{xx}+u_{yy})(x,y) = x^2+y^2$$

and u is the solution of the boundary value problem.

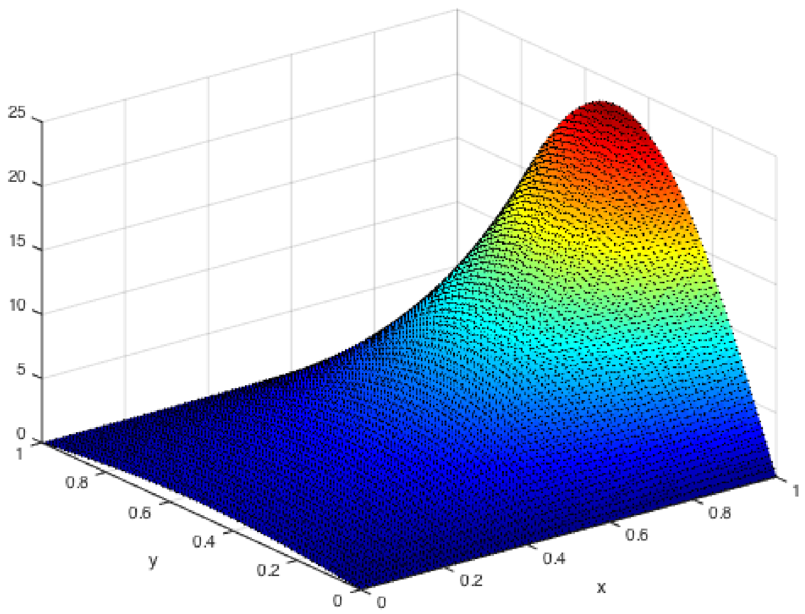


Figure 0.1: The solution $u$ on the domain $\Omega = [0;1] \times [0;1]$

**Exercise 2:**
In the second exercise we want to compare the solution of a Poisson problem with its approximated solution via 5-point-stencil.

$$\begin{aligned} &\textbf{Continuous problem} && \textbf{Discrete problem} \\ \Delta u &= f \text{ on } \Omega & \Delta_h u_h &= f, \text{ on } \Omega_h \\ u &= g \text{ on } \partial\Omega & u_h &= g, \text{ on } \partial\Omega_h \end{aligned} \tag{0.5}$$

- We start with the discrete maximum principle and show that a function $v_h$ with $\Delta_h v_h \geq 0$ attains its maximum at the border. Recall that we define the "discrete Laplacian" via the five-point-stencil from Exercise 1,

$$\Delta_h v_h(x,y) = \frac{1}{h^2}(v_h(x+h,y) + v_h(x,y+h) + v_h(x-h,y) + v_h(x,y-h) - 4v_h(x,y)) \geq 0 \tag{0.6}$$

for all inner points $(x,y) \in \Omega_h$. We can rewrite this inequality in the form

$$v_h(x,y) \leq \frac{1}{4}(v_h(x+h,y) + v_h(x,y+h) + v_h(x-h,y) + v_h(x,y-h)),$$

so the value at any inner point is bounded from above by the mean of the four points surrounding it. Naturally, at least one of the four values $v_h(x+h,y)$, $v_h(x,y+h)$, $v_h(x-h,y)$ or $v_h(x,y-h)$ has to be as large as $v_h(x,y)$. Hence, $v_h(x,y)$ can not be the global maximum. We can also proof this observation formally by contradiction. Assume that $(x^*,y^*) \in \Omega_h$ is an inner point such that

$$v_h(x^*,y^*) > v_h(x,y) \quad \forall\ (x,y) \in \Omega_h \cup \partial\Omega_h.$$

Then, in particular, $v_h(x^*,y^*) > v_h(x^*+h,y^*)$, $v_h(x^*,y^*) > v_h(x^*-h,y^*)$ etc., so

$$4v_h(x^*,y^*) > v_h(x^*+h,y^*) + v_h(x^*,y^*+h) + v_h(x^*-h,y^*) + v_h(x^*,y^*-h).$$

This is equivalent to $\Delta_h v_h(x^*,y^*) < 0$ which is a contradiction to our primary assumption.

- The proof follows completely analogous to the previous proof. Assume that $v_h$ has an absolute minimum at $(x^*, y^*) \in \Omega_h$, i.e.

$$v_h(x^*, y^*) < v_h(x, y) \quad \forall\ (x, y) \in \Omega_h \cup \partial\Omega_h.$$

Then,

$$v_h(x^*, y^*) < \frac{1}{4}(v_h(x^* + h, y^*) + v_h(x^*, y^* + h) + v_h(x^* - h, y^*) + v_h(x^*, y^* - h)).$$

This again implies $\Delta_h v_h(x^*, y^*) > 0$, what is a contradiction to $\Delta_h v_h(x, y) \le 0$ for all $(x, y) \in \Omega_h$. So, the global minimum can not be attained in $\Omega_h$.

- To show uniqueness of the solution, we use the hint and suppose that there are two solutions $u_h^1$ and $u_h^2$ with

$$\begin{aligned} \Delta_h u_h^1 &= f \text{ on } \Omega_h & \qquad \Delta_h u_h^2 &= f \text{ on } \Omega_h, \\ u_h^1 &= g \text{ on } \partial\Omega_h & u_h^2 &= g \text{ on } \partial\Omega_h. \end{aligned} \tag{0.7}$$

We have to show that $u_h^1 - u_h^2 \equiv 0$. We start with the Poisson problem for $u_h^1 - u_h^2$. We know that the difference $u_h^1 - u_h^2$ satisfies

$$\begin{aligned} \Delta_h(u_h^1 - u_h^2) &= 0 \text{ on } \Omega_h, \\ u_h^1 - u_h^2 &= 0 \text{ on } \partial\Omega_h. \end{aligned}$$

Due to both previous results, we know that $u_h^1 - u_h^2$ attain its minimum and its maximum at the boundary $\partial\Omega_h$. But since $u_h^1 - u_h^2 = 0$ at the boundary, it follows that

$$u_h^1 - u_h^2 \equiv 0 \text{ on } \Omega_h.$$

So, the solution to the boundary value problem is unique.

- So far, we have shown that the five-stencil-approximation has a unique solution. Next, we want to give an estimate for this solution based on $f$ and $g$. Again, we start with the hint and consider the function

$$u_h = v_h + M_f\phi = v_h + \max_{\Omega_h} |f|\phi$$

with $\phi : \mathbb{R}^2 \mapsto \mathbb{R}, (x, y) \mapsto \frac{1}{4}\left[(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2\right]$. To estimate $\Delta v_h$ we calculate the Laplacian of $u_h$,

$$\Delta u_h = \Delta v_h + M_f = f + M_f$$

with $\phi_{xx} = \frac{1}{2}$ and $\phi_{yy} = \frac{1}{2}$. To use again the maximum principle, we require

$$\Delta u_h \ge 0,$$

but this holds true since $M_f = \max_{\Omega_h} |f| \ge f$. So, $\max_{\Omega_h} u_h \le \max_{\partial\Omega_h} u_h$ and it suffices to evaluate $u_h$ at the boundary. We have

$$u_h = g + M_f\phi$$

at $\partial\Omega_h$ and thus,

$$\max_{\Omega_h} u_h \le \max_{\partial\Omega_h} u_h \le \max_{\partial\Omega_h} |g| + \frac{1}{8}M_f.$$

We used that $\phi$ attains its maximum on the unit square at the vertices $\phi(0, 0) = \phi(1, 0) = \phi(0, 1) = \phi(1, 1) = \frac{1}{8}$.

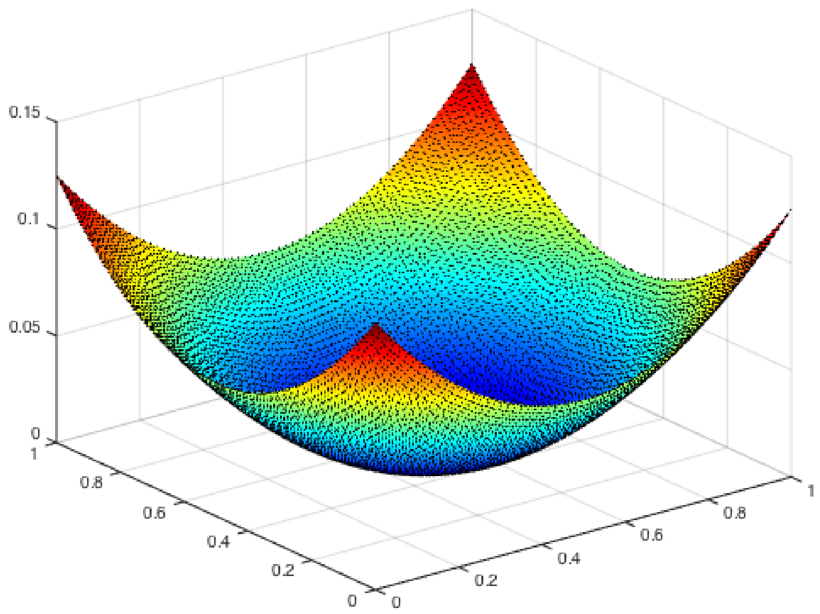


Figure 0.2: $\phi$ on the domain $\Omega = [0;1] \times [0;1]$

- As a last step we want to give an estimate for the error $|u - u_h|$ and show that the approximated solution $u_h$ converges to $u$ as $h \to 0$.
  We extend the previous estimate to make a statement about the accuracy of the approximated solution.
  Let $u$ denote the solution of the continuous problem, i.e.

$$\Delta u = f \text{ on } \Omega, \quad u = g \text{ on } \partial\Omega, \tag{0.8}$$

while $u_h$ is the solution to the discrete problem

$$\Delta_h u_h = f \text{ on } \Omega_h, \quad u_h = g \text{ on } \partial\Omega_h. \tag{0.9}$$

Then, the difference $u - u_h$ satisfies

$$\Delta_h(u - u_h) = K \text{ on } \Omega_h, \quad u - u_h = 0 \text{ on } \partial\Omega_h \tag{0.10}$$

with $K = \Delta_h u - \Delta_h u_h = \Delta_h u - f = \Delta_h u - \Delta u$. With Equation (3) from the worksheet, we conclude

$$\max_{\Omega_h} |u - u_h| \leq \frac{1}{8} \max_{\Omega_h} |K| = \frac{1}{8} \max_{\Omega_h} |\Delta_h u - \Delta u|.$$

- Since the five-point-stencil is an method of accuracy $\mathcal{O}(h^2)$, we have

$$|\Delta_h u - \Delta u| \leq Ch^2 \tag{0.11}$$

by definition. Thus, we get

$$\max_{\Omega_h} |u - u_h| \leq Ch^2$$

and taking the limit $h \to 0$ leads to

$$\lim_{h\to 0} |u - u_h| \leq \lim_{h\to 0} \max_{\Omega_h} |u - u_h| \leq 0.$$

Since we take the limit of the absolute value, this is equal to $\lim_{h\to 0} |u - u_h| = 0$.