Solutions to Worksheet 6

Exercise 2:
The formal Fourier series of a function $f \in L^2([0;1])$ is given by

$$f(x) = \sum_{k\in\mathbb{Z}} c_k e^{2\pi ikx}. \tag{0.1}$$

To approximate the solution of the differential equation

$$y''(x) + y(x) = e^{\sin(2\pi x)} \tag{0.2}$$

with the FFT we use as ansatz

$$y(x) = \sum_{-N}^{N} c_k e^{2\pi ikx}. \tag{0.3}$$

for some $N \in \mathbb{N}$. For the second derivative of y, this means

$$y'(x) = \sum_{-N}^{N} c_k \cdot (2\pi ik) e^{2\pi ikx},$$
$$y''(x) = \sum_{-N}^{N} c_k \cdot (2\pi ik)^2 e^{2\pi ikx}.$$

For the left hand side of Equation (0.2), we hence have

$$y''(x) + y(x) = \sum_{-N}^{N} (1 - (2\pi k)^2) c_k \cdot e^{2\pi ikx}.$$

Accordingly, we can also write the right hand side as Fourier series

$$e^{\sin(2\pi x)} = \sum_{-N}^{N} d_k e^{2\pi ikx}, \tag{0.4}$$

where the coefficients d_k are given by $d_k = \langle e^{\sin(2\pi x)}, e^{2\pi ikx}\rangle$. To observe this fact, note that the set $\{e^{2\pi ikx}\}_{k\in\mathbb{Z}}$ forms an orthonormal basis of the space $L^2([0;1])$. For $\ell \neq k$:

$$\langle e^{2\pi i\ell x}, e^{2\pi ikx}\rangle_{L^2} = \int_0^1 e^{2\pi i(\ell-k)x}\ dx = \frac{1}{2\pi i(\ell-k)}(e^{2\pi i(\ell-k)} - 1) = 0$$

since $e^{2\pi in} = 1$ for all $n \in \mathbb{N}$. For $\ell = k$ it is

$$\|e^{2\pi ikx}\|_{L^2}^2 = \langle e^{2\pi ikx}, e^{2\pi ikx}\rangle_{L^2} = \int_0^1 1\ dx = 1.$$

So, $\langle e^{2\pi i\ell x}, e^{2\pi ikx}\rangle_{L^2} = \delta_{\ell,k}$ and therefore, for a Fourier series as defined in (0.1), it holds

$$\langle f, e^{2\pi ikx}\rangle_{L^2} = \langle \sum_{j\in\mathbb{Z}} c_j e^{2\pi ijx}, e^{2\pi ikx}\rangle_{L^2} = \sum_{j\in\mathbb{Z}} c_j \langle e^{2\pi ijx}, e^{2\pi ikx}\rangle_{L^2} = c_k. \tag{0.5}$$

We see that our coefficients d_k for the right hand side coincide with the values a_k that are calculated with the FFT. Moreover from $y''(x) + y(x) = e^{\sin(2\pi x)}$ it follows that

$$\sum_{-N}^{N} (1 - (2\pi k)^2) c_k \cdot e^{2\pi ikx} = \sum_{-N}^{N} d_k e^{2\pi ikx} \tag{0.6}$$

and thus, we can deduce the coefficients c_k from

$$(1-(2\pi k)^2)c_k = d_k \quad \text{for } -N \leq k \leq N. \tag{0.7}$$

Inserting this into the ansatz (0.3) yields the approximated solution.

Exercise 3:
An ODE of the present form can be solved by **variation of parameters**. This principle works as follows:
First, we solve the homogenous ODE

$$y'(t) = -\frac{1}{\varepsilon}y(t). \tag{0.8}$$

One could see that the solution to this equation is $y(t) = c_0 e^{-\frac{1}{\varepsilon}t}$ or use separation of variables

$$\begin{aligned}
\frac{dy}{dt} &= -\frac{1}{\varepsilon}y,\\
\frac{dy}{y} &= -\frac{1}{\varepsilon}dt,\\
\int \frac{1}{y}\, dy &= -\int \frac{1}{\varepsilon}\, dt,\\
\ln|y| &= -\frac{1}{\varepsilon}t + C,\\
y &= \pm e^{-\frac{1}{\varepsilon}t+C} := c_0 e^{-\frac{1}{\varepsilon}t}.
\end{aligned}$$

As next step we consider c_0 as a differentiable function $c_0(t)$ and determine it such that the inhomogenous ODE also holds, i.e.

$$y(t) = c_0(t)e^{-\frac{1}{\varepsilon}t} \text{ satisfies } y'(t) = -\frac{1}{\varepsilon}(y(t)-\sin(t)),\ y(0)=1. \tag{0.9}$$

We start with the derivative of $y(t)$.

$$y'(t) = c_0'(t)e^{-\frac{1}{\varepsilon}t} - \frac{1}{\varepsilon}c_0(t)e^{-\frac{1}{\varepsilon}t} = -\frac{1}{\varepsilon}(y(t) - \varepsilon c_0'(t)e^{-\frac{1}{\varepsilon}t})$$

Comparing the terms shows that $\varepsilon c_0'(t)e^{-\frac{1}{\varepsilon}t}$ must be equal to $\sin(t)$. This, again, gives us an ODE with the initial value $c_0(0) = 1$, since

$$y(0) = c_0(0)\cdot 1 \overset{!}{=} 1. \tag{0.10}$$

Thus,

$$\begin{aligned}
\varepsilon c_0'(t)e^{-\frac{1}{\varepsilon}t} &= \sin(t),\\
c_0'(t) &= \frac{1}{\varepsilon}\sin(t)e^{\frac{1}{\varepsilon}t},\\
c_0(t) &= \frac{1}{\varepsilon}\int \sin(t)e^{\frac{1}{\varepsilon}t}\, dt + c_1.
\end{aligned}$$

Integrals of this type can be solved by applying two times integration by parts (recall: $\int u'(x)v(x)\, dx = u(x)v(x) - \int u(x)v'(x)\, dx$). We have

$$\begin{aligned}
\int \sin(t)e^{\frac{1}{\varepsilon}t}\, dt &= -\cos(t)e^{\frac{1}{\varepsilon}t} + \frac{1}{\varepsilon}\int \cos(t)e^{\frac{1}{\varepsilon}t}\, dt\\
&= -\cos(t)e^{\frac{1}{\varepsilon}t} + \frac{1}{\varepsilon}\left(\sin(t)e^{\frac{1}{\varepsilon}t} - \frac{1}{\varepsilon}\int \sin(t)e^{\frac{1}{\varepsilon}t}\, dt\right).
\end{aligned}$$

So,

$$
\begin{aligned}
(1+\frac{1}{\varepsilon^2})\int \sin(t)e^{\frac{1}{\varepsilon}t}\ dt &= (\frac{1}{\varepsilon}\sin(t)-\cos(t))e^{\frac{1}{\varepsilon}t},\\
\int \sin(t)e^{\frac{1}{\varepsilon}t}\ dt &= \frac{\varepsilon^2}{1+\varepsilon^2}(\frac{1}{\varepsilon}\sin(t)-\cos(t))e^{\frac{1}{\varepsilon}t},\\
\int \sin(t)e^{\frac{1}{\varepsilon}t}\ dt &= \frac{\varepsilon}{1+\varepsilon^2}(\sin(t)-\varepsilon\cos(t))e^{\frac{1}{\varepsilon}t}
\end{aligned}
$$

and

$$
c_0(t) = \frac{1}{1+\varepsilon^2}(\sin(t)-\varepsilon\cos(t))e^{\frac{1}{\varepsilon}t} + c_1. \tag{0.11}
$$

It remains to identify c_1 with the intial value $c_0(0) = 1$.

$$
\begin{aligned}
c_0(0) = -\frac{\varepsilon}{1+\varepsilon^2} + c_1 &\overset{!}{=} 1\\
c_1 &= 1 + \frac{\varepsilon}{1+\varepsilon^2}.
\end{aligned}
$$

All in all, the solution to the given ODE reads

$$
\begin{aligned}
y(t) &= c_0(t)e^{-\frac{1}{\varepsilon}t} = (\frac{1}{1+\varepsilon^2}(\sin(t)-\varepsilon\cos(t))e^{\frac{1}{\varepsilon}t} + 1 + \frac{\varepsilon}{1+\varepsilon^2})e^{-\frac{1}{\varepsilon}t}\\
&= \frac{1}{1+\varepsilon^2}(\sin(t)-\varepsilon\cos(t)) + (1+\frac{\varepsilon}{1+\varepsilon^2})e^{-\frac{1}{\varepsilon}t}.
\end{aligned}
$$