# Solutions to Worksheet 2

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**Exercise 1:**

Partial differential equations of order 2 can be distinguished into three different basic types: elliptic, parabolic and hyperbolic PDEs. Any second order PDE can be written in the form

$$au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = 0, \tag{0.1}$$

where $a, ..., f \in \mathbb{R}$ are coefficients or coefficient functions, e.g. $a : \mathbb{R}^2 \mapsto \mathbb{R}, a = a(x, y)$ et cetera. For the classification of the PDE, only the prefactors of the second derivatives are taken into account. The discriminant of the PDE is defined as

$$D = b^2 - ac. \tag{0.2}$$

(See also conic sections, quadratic equations.)
If $D < 0$, the PDE is called elliptic, if $D = 0$ it is parabolic and the PDE is called hyperbolic for $D > 0$.
Note that the discriminant can depend on $x$ and $y$, $D = D(x, y)$ and thus, the PDE can attain different types in different regions.

- $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$
  Here, $a = 9$, $b = 3$ and $c = 1$. So $D = 9 - 9 = 0$, the PDE is **parabolic**.
- $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0$
  We have $a = 1$ and $c = 2$. Since $u_{xy} = u_{yx}$, moreover $b = -2$ and $D = 4 - 2 = 2 > 0$. The PDE is **hyperbolic**.
- $(x^2 - 1)u_{xx} - xu_x + 2xyu_{xy} + (y^2 - 1)u_{yy} + yu_y = 0$
  In this example we have coefficient functions, $a(x, y) = x^2 - 1$, $b(x, y) = xy$ and $c(x, y) = y^2 - 1$. Thus, $D(x, y) = x^2y^2 - (x^2 - 1)(y^2 - 1) = x^2 + y^2 - 1$.
  - $D(x, y) = 0$ if $x^2 + y^2 = 1$. So, the PDE is **parabolic** on the unit circle.
  - $D(x, y) < 0$ if $x^2 + y^2 < 1$. So, the PDE is **elliptic** on the open unit disk.
  - $D(x, y) > 0$ if $x^2 + y^2 > 1$. So, the PDE is **hyperbolic** outside of the closed unit disk.

**Exercise 2:**

- It is easy to see that the Gram matrix is symmetric since $\nabla\phi_i(x)^T\nabla\phi_j(x) = \nabla\phi_j(x)^T\nabla\phi_i(x)$, so $G_{i,j} = G_{j,i}$.
  $G$ is positive definite if for all $t \in \mathbb{R}^N\backslash\{0\}$

  $$t^TGt > 0. \tag{0.3}$$

  The definition of $G_{i,j}$ is the definition of an inner product

  $$\langle\phi_i, \phi_j\rangle_\Omega := \int_\Omega \nabla\phi_i^T\nabla\phi_j dx. \tag{0.4}$$

Recall that an inner product is a bilinear, symmetric function i.e.

$$\alpha\langle\phi,\psi\rangle = \langle\alpha\phi,\psi\rangle = \langle\phi,\alpha\psi\rangle, \tag{0.5}$$
$$\langle\phi_1,\psi\rangle + \langle\phi_2,\psi\rangle = \langle\phi_1+\phi_2,\psi\rangle. \tag{0.6}$$

So let $t \in \mathbb{R}^N$, then

$$t^T G t = \sum_{i,j=1}^{N} t_i \langle\phi_i,\phi_j\rangle_\Omega t_j = \sum_{i,j=1}^{N} \langle t_i\phi_i, t_j\phi_j\rangle_\Omega \tag{0.7}$$
$$= \langle\sum_{i=1}^{N} t_i\phi_i, \sum_{j=1}^{N} t_j\phi_j\rangle_\Omega = \langle\sum_{i=1}^{N} t_i\phi_i, \sum_{i=1}^{N} t_i\phi_i\rangle_\Omega \tag{0.8}$$
$$= \|\sum_{i=1}^{N} t_i\phi_i\|^2 \geq 0. \tag{0.9}$$

It remains to prove that $\|\sum_{i=1}^{N} t_i\phi_i\|^2 = 0 \Leftrightarrow t = 0$, but this is a direct consequence of the linear independence of the $\{\phi_i\}_{i=1}^N$:

$$\|\sum_{i=1}^{N} t_i\phi_i\|^2 = 0 \quad \Leftrightarrow \quad \sum_{i=1}^{N} t_i\phi_i = 0 \quad \overset{\text{ONB}}{\Leftrightarrow} \quad t_i = 0\ \forall\ i = 1,...,N \quad \Leftrightarrow \quad t = 0 \tag{0.10}$$

- We are given the one-dimensional boundary value problem

$$-\frac{d}{dx}(a(x)\frac{d}{dx}u(x)) + b(x)u(x) = f(x), \quad u(a) = u_a,\ u(b) = u_b \text{ and } a \leq x \leq b. \tag{0.11}$$

To get the weak formulation of this problem we introduce test functions $v$ defined on the domain $I = [a;b]$ satisfying $v(a) = v(b) = 0$. Moreover, we define the usual $L^2$-inner product on $I$,

$$\langle\phi,\psi\rangle_{L^2(I)} = \int_a^b \phi(x)\psi(x)\ dx \tag{0.12}$$

for functions $\phi, \psi : I \mapsto \mathbb{R}$. In general, to obtain the weak formulation of a PDE of the form $Au = f$ we take the space of all test function $V$ and look for an $u$ that satisfies

$$\langle Au, v\rangle = \langle f, v\rangle \tag{0.13}$$

for all test functions $v \in V$. Here, taking the inner product leads to

$$\int_a^b (-\frac{d}{dx}(a(x)\frac{d}{dx}u(x)) + b(x)u(x))v(x)dx = \int_a^b f(x)v(x)dx, \tag{0.14}$$
$$-\int_a^b \frac{d}{dx}(a(x)\frac{d}{dx}u(x))v(x)dx + \int_a^b b(x)u(x)v(x)dx = \int_a^b f(x)v(x)dx, \tag{0.15}$$
$$\tag{0.16}$$

We can use partial integration to simplify the first integral

$$\int_a^b \frac{d}{dx}(a(x)\frac{d}{dx}u(x))v(x)dx = \left[a(x)\frac{d}{dx}u(x)v(x)\right]_a^b - \int_a^b a(x)\frac{d}{dx}u(x)\frac{d}{dx}v(x)dx \tag{0.17}$$
$$= -\int_a^b a(x)\frac{d}{dx}u(x)\frac{d}{dx}v(x)dx. \tag{0.18}$$

Hence, the weak formulation of the problem is

$$\int_a^b a(x)\frac{d}{dx}u(x)\frac{d}{dx}v(x) + b(x)u(x)v(x)dx = \int_a^b f(x)v(x)dx. \tag{0.19}$$

A suitable test space in this case would be $H_0^1(I)$. Then, taking the integral of $v$ and the derivative of $v$ is well-defined.

**Exercise 3:**

- We are given two Hamiltonian flows satisfying

$$\dot{\Phi}^{G,t}(z) = \Omega\nabla G(\Phi^{G,t}(z)), \quad \Phi^{G,0}(z) = z \quad \text{and} \quad \dot{\Phi}^{H,t}(z) = \Omega\nabla H(\Phi^{H,t}(z)), \quad \Phi^{H,0}(z) = z. \tag{0.20}$$

We have to show that

$$G(\Phi^{H,t}(z)) = G(z), \quad \forall t, z \quad \Leftrightarrow \quad H(\Phi^{G,t}(z)) = H(z), \quad \forall t, z. \tag{0.21}$$

Due to the initial condition $\Phi^{\cdot,0}(z) = z$ both equations hold true for $t = 0$. So, to proof that they stay satisfied for all times $t \geq 0$ we have to show that $\partial_t G(\Phi^{H,t}(z)) = 0$ resp. $\partial_t H(\Phi^{G,t}(z)) = 0$.
Using the chain rule the derivatives are

$$\partial_t G(\Phi^{H,t}(z)) = \dot{\Phi}^{H,t}(z)^T\nabla G(\Phi^{H,t}(z)) = \nabla H(\Phi^{H,t}(z))^T\Omega^T\nabla G(\Phi^{H,t}(z)), \tag{0.22}$$
$$\partial_t H(\Phi^{G,t}(z)) = \dot{\Phi}^{G,t}(z)^T\nabla H(\Phi^{G,t}(z)) = \nabla G(\Phi^{G,t}(z))^T\Omega^T\nabla H(\Phi^{G,t}(z)). \tag{0.23}$$

To make things clearer we evaluate both expressions at $z(t)$. Since they are scalars we are furthermore allowed to transpose them.

$$\partial_t G(z(t)) = \nabla H(z(t))^T\Omega^T\nabla G(z(t)) = -\nabla H(z(t))^T\Omega\nabla G(z(t)), \tag{0.24}$$
$$\partial_t H(z(t)) = \nabla G(z(t))^T\Omega^T\nabla H(z(t)) = \nabla H(z(t))^T\Omega\nabla G(z(t)). \tag{0.25}$$

Thus, $\partial_t G(z(t)) = -\partial_t H(z(t))$ and in particular $\partial_t G = 0 \;\Leftrightarrow\; \partial_t H = 0$.

- We use the previous results and show instead of $G(\Phi^{H,t}(z)) = G(z)$ that $H(\Phi^{G,t}(z)) = H(z)$ holds true. To derive the Hamiltonian flow of $G$ we need its gradient

$$\nabla_z G = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}. \tag{0.26}$$

Then, the solution of the corresponding Hamiltonian system

$$\dot{z}(t) = \Omega\nabla_z G, \quad z(0) = z_0 \tag{0.27}$$

reads

$$z(t) = z(0) + t \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}. \tag{0.28}$$

Inserting in the definition of $H$ yields the conservation, $H(z(t)) = H(z(0))$.