MOCK EXAM: MONTE CARLO METHODS
WINTER TERM 2015/16

EXAMINER: CAROLINE LASSER

The exam has 5 items with 8 points per item.

The examination time is 90 minutes.

Please answer each question on its sheet. If necessary, write on its reverse side.

Name:

Matriculation number:

Degree programme:

1	2	3	4	5

Sum:

Date: 12. Januar 2016.

1. Basic notions

Please provide short descriptions respectively definitions. 2–3 precise sentences per topic suffice.

(1) Direct simulation

(2) Importance sampling

(3) Dobrushin's ergodicity coefficient

(4) Detailed balance

2. Basic methods

Consider the density

$$f : \mathbb{R} \to [0, \infty[, \quad x \mapsto \mu e^{-\mu x} \chi_{[0,\infty[}(x)$$

of the exponential distribution with parameter $\mu > 0$.

(1) Calculate the cumulative distribution function F and

$$F^{-1}(u) = \inf\{x \in \mathbb{R} : F(x) = u\}, \quad u \in]0, 1[,$$

for the exponential distribution with parameter μ.

(2) Let U be a random variable uniformly distributed on $[0, 1]$. Prove that $F^{-1}(U)$ is exponentially distributed with parameter μ.

(3) Provide MATLAB type pseudocode for the direct simulation of

$$a = \int_{\mathbb{R}} x f(x) dx$$

based on n samples.

3. Algorithms

Consider the transition matrix of a Metropolis chain,

$$\pi(x,y) = \begin{cases} G(x,y)\exp(-(H(y)-H(x))^+), & x \neq y, \\ 1 - \sum_{z\neq y} \pi(x,z), & x = y, \end{cases}$$

for a given stochastic $n \times n$ matrix G and $H : \{1,\dots,n\} \to \mathbb{R}$.

(1) What does the Metropolis algorithm aim at? Please answer in 1–2 precise sentences.

(2) Describe the steps of the corresponding Metropolis algorithm in 2-3 precisesentences.

(3) Explain in 1–2 precise sentences, why the Metropolis chain may escape local energy minima.

4. Evaluating

(1) Explain in 2–3 precise sentences, why variance reduction is important for direct simulation.

(2) Describe in 2–3 precise sentences the method of antithetic sampling.

(3) Explain in 1–2 precise sentences. why antithetic sampling may reduce variance.

5. Proofing

(1) Consider the Metropolis sampler π with symmetric proposal matrix. Prove that π is reversible with respect to the Gibbs field.

(2) Consider a Gibbs field on a finite space X for inverse temperature $\beta > 0$. Calculate the cooling limit $\beta \to \infty$ of the Gibbs field.