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Monte Carlo Methods

Worksheet 7: Markov Chains

Exercise 17 (The hard core model)**.** Consider the two-dimensional grid $G = \{1, \dots, m\}^2$ and the state space

$$Z = \{z : G \to \{0,1\} \mid z(x) = z(y) = 1 \Rightarrow \|x - y\|_1 \neq 1\}.$$

a) Simulate the uniform distribution on Z by the rejection method, that is, drawing a vector z of m^2 independent Bernoulli random numbers and accepting if $z \in Z$. Let $m = 5$. What is the acceptance rate for $n = 10^3$ independent repetitions? Answer the same question for $m = 6$.

b) Consider the transition matrix

$$p_{z,z'} = \begin{cases} 1/|G|, & \|z - z'\|_1 = 1, \\ 0, & \|z - z'\|_1 > 1, \\ 1 - \sum_{w \neq z} p_{z,w}, & z = z', \end{cases}$$

for $z, z' \in Z$ and compute the ergodicity coefficient $c(P)$. Prove that the uniform distribution μ on Z is stationary with respect to P.

c) Consider the function

$$f(z) = \frac{1}{|G|} \sum_{x \in G} z(x), \qquad z \in Z,$$

and show that $\mathbb{E}_\mu(f) = \frac{1}{|Z|} \sum_{z \in Z} f(z)$.

d) Start a Markov chain $(X_n)_{n \geq 0}$ with transition matrix P and initial checkerboard distribution. Simulate

$$M_n(f) = \frac{1}{n} \sum_{i=1}^{n} f(X_i), \qquad n = 10^2, \dots, 10^6,$$

for $m = 5, 8, 32, 128$. Plot your results as functions of n. Does 0.25 emerge as limiting value?