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Monte Carlo Methods

Worksheet 6: Countable Markov Chains

Exercise 15 (Ergodicity coefficient)**.** Let P be a stochastic matrix for a countable index set E and denote its row vectors by p_i, $i \in E$. Prove that Dobrushin's ergodicity coefficient

$$c(P) := \sup \left\{ \|w^T P\|_1 / \|w\|_1 : w \neq 0, \sum_{i \in E} w_i = 0 \right\}$$

satisfies

$$c(P) = \tfrac{1}{2} \sup_{i,j \in E} \|p_i - p_j\|_1$$

by first proving the following two auxiliary results:

a) For all $w \in \ell^1(E)$ it holds $\|w\|_1 = \sup\{|w^T x| : \|x\|_\infty = 1\}$.

b) For all $w, x \in \ell^1(E)$ with $\sum_{i \in E} w_i = 0$ one has

$$|w^T x| \leq \tfrac{1}{2} \sup_{i,j \in E} |x_i - x_j| \cdot \|w\|_1.$$

Exercise 16 (Random walk)**.** Let $p \in]0,1[$ and $(Z_n)_{n \geq 0}$ be an i.i.d. sequence with

$$P(Z_n = 1) = p, \qquad P(Z_n = -1) = 1 - p.$$

Set $X_0 = 0$ and $X_{n+1} = (X_n + Z_n)$ for $n \geq 0$.

a) Prove that $(X_n)_{n \geq 0}$ is a homogeneous Markov chain. Derive its transition matrix P and compute its ergodicity coefficient $c(P)$.

b) Simulate and plot random trajectories of length $N = 100$ and $N = 10^5$ for the symmetric $p = \frac{1}{2}$ random walk. Also plot the two corresponding histograms.

c) Simulate and plot random trajectories of length $N = 100$ and $N = 10^5$ for the asymmetric $p = \frac{1}{3}$ random walk. Also plot the two corresponding histograms.