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Monte Carlo Methods

Worksheet 1: Direct Simulation

Exercise 1 (Integrability). We consider the functions

$$f : [0,1] \to \mathbb{R},\ x \mapsto x^{-1/4}, \qquad g : [0,1] \to \mathbb{R},\ x \mapsto x^{-3/4}.$$

a) Compute $I(f)$ and $I(g)$ by hand.
b) Compute $I(f)$ and $I(g)$ by direct simulation with n pseudorandom samples for $n \in [10, 10^6]$.
c) Plot your simulation results as functions of n. Can they be related with $1/\sqrt{n}$?

Exercise 2 (The unit ball). Let X be uniformly distributed on $[0,1]^d$. Let $Y = \chi_B(X)$ for the unit ball $B = \{x \in \mathbb{R}^d : |x| \le 1\}$.

a) Compute $\mathbb{E}(Y)$ and $\mathbb{V}(Y)$ by hand.
b) Compute $\mathbb{E}(Y)$ by direct simulation for $d = 3$ and $d = 6$ using n pseudorandom samples for $n \in [10, 10^6]$.
c) Plot your simulation results as functions of n. Do they depend on d?

Exercise 3 (Variance estimation). Let X be uniformly distributed on $[0,1]$ and $Y = f(X)$ for the function f of Exercise 1.

a) Compute $\sigma(Y)$ by hand.
b) Compute for your simulation results of Exercise 1 the variance estimator V_n for $n \in [10, 10^6]$ and plot V_n as a function of n.

Exercise 4 (Confidence intervals). Let $a \in \mathbb{R}$, and denote by D_n the random variable for the direct simulation of a with n samples. If L_n is a positive random variable such that the interval $I_n(\omega) = [D_n(\omega) - L_n(\omega), D_n(\omega) + L_n(\omega)]$, $\omega \in \Omega$, satisfies

$$P(a \in I_n) \ge 1 - \delta,$$

then I_n is called a *confidence interval* for a with level $1 - \delta$, where $\delta \in (0,1)$.

Consider Exercise 1 for $n = 5000$ samples and simulate 100 different intervals I_n with $\delta = 0.05$ for the following two choices of L_n.

a) $L_n = \sqrt{1/(2n)\ln(2/\delta)}$,
b) $L_n = \Phi^{-1}(1 - \delta/2)\sqrt{V_n/n}$, where Φ is the cumulative distribution function of the standard normal distribution and V_n the empirical variance.

Visualize and discuss your results.