Preface

This book will take you on a thrilling tour of some of the most important and powerful areas of contemporary numerical mathematics. A first unusual feature is that the tour is organized by problems, not methods: it is extremely valuable to realize that numerical problems often yield to a wide variety of methods. For example, we solve a random-walk problem (Chapter 6) by several different techniques, such as large-scale linear algebra, a three-term recursion obtained by symbolic computations, elliptic integrals and the arithmetic-geometric mean, and Fourier analysis. We do so in IEEE arithmetic to full accuracy and, at the extreme, in high-precision arithmetic to get 10,000 digits.

A second unusual feature is that we very carefully try to justify the validity of every single digit of a numerical answer, using methods ranging from carefully designed computer experiments and a posteriori error estimates to computer-assisted proofs based on interval arithmetic. In the real world, the first two methods are usually adequate and give the desired confidence in the answer. Interval methods, while nicely rigorous, would most often not provide any additional benefit. Yet it sometimes happens that one of the best approaches to a problem is one that provides proof along the way (this occurs in Chapter 4), a point that has considerable mathematical interest.

A main theme of the book is that there are usually two options for solving a numerical problem: either use a brute force method running overnight and unsupervised on a high-performance workstation with lots of memory, or spend your days thinking harder, with the help of mathematical theory and a good library, in the hope of coming up with a clever method that will solve the problem in less than a second on common hardware. Of course, in practice these two options of attacking a problem will scale differently with problem size and difficulty, and your choice will depend on such resources as your time, interest, and knowledge and the computer power at your disposal. One noteworthy case, where a detour guided by theory leads to an approach that is ultimately much more efficient than the direct path, is illustrated on the cover of this book. That diagram (taken from Chapter 1) illustrates that many problems about real numbers can be made much, much easier by stepping outside the real axis and taking a route through the complex plane.

The waypoints of our tour are the 10 problems published in the January/February 2002 issue of *SIAM News* by Nick Trefethen of Oxford University as an intriguing computing challenge to the mathematical public. The answer to each problem is a real number; entrants had to compute several digits of the answer. Scoring was simple: 1 point per digit, up to a maximum of 10 per problem. Thus a perfect score would be 100. When the dust settled several months later, entries had been received from 94 teams in 25 countries. Twenty of those teams achieved a perfect score of 100 and 5 others got 99 points. The whole fascinating story, including the names of the winners, is told in our introductory chapter, "The Story."

The contest, now known as the *SIAM 100-Digit Challenge*, was noteworthy for several reasons. The problems were quite diverse, so while an expert in a particular field might have little trouble with one or two of them, he or she would have to invest a lot of time to learn enough about the other problems to solve them. Only digits were required, neither proofs of existence and uniqueness of the solution, convergence of the method, nor correctness of the result; nevertheless, a serious team would want to put some effort into theoretical investigations. The impact of modern software on these sorts of problems is immense, and it is very useful to try all the major software tools on these problems so as to learn their strengths and their limitations.

This book is written at a level suitable for beginning graduate students and could serve as a text for a seminar or as a source for projects. Indeed, these problems were originally assigned in a first-year graduate course at Oxford University, where they were used to challenge students to think beyond the basic numerical techniques. We have tried to show the diversity of mathematical and algorithmic tools that might come into play when faced with these, and similar, numerical challenges, such as:

- large-scale linear algebra
- computational complex analysis
- special functions and the arithmetic-geometric mean
- Fourier analysis
- asymptotic expansions
- convergence acceleration
- discretizations that converge exponentially fast
- symbolic computing
- global optimization
- Monte Carlo and evolutionary algorithms
- chaos and shadowing
- stability and accuracy
- a priori and a posteriori error analysis
- high-precision, significance, and interval arithmetic

We hope to encourage the reader to take a broad view of mathematics, since one moral of this contest is that overspecialization will provide too narrow a view for one with a serious interest in computation.

The chapters on the 10 problems may be read independently. Because convergence acceleration plays an important role in many of the problems, we have included a discussion of the basic methods in Appendix A. In Appendix B we summarize our efforts in calculating the solutions to extremely high accuracies. Appendix C contains code that solves the 10 problems in a variety of computing environments. We have also included in Appendix D a sampling of additional problems that will serve as interesting challenges for readers who have mastered some of the techniques in the book.

All code in the book as well as some additional material related to this project can be found at an accompanying web page:

`www.siam.org/books/100digitchallenge`

We four authors, from four countries and three continents, did not know each other prior to the contest and came together via e-mail to propose and write this book. It took thousands of e-mail messages and exchanges of files, code, and data. This collaboration has been an unexpected benefit of our participation in the *SIAM 100-Digit Challenge.*

Notation and Terminology. When two real numbers are said to agree to d digits, one must be clear on what is meant. In this book we ignore rounding and consider it a problem of strings: the two strings one gets by truncating to the first d significant digits have to be identical. We use the symbol $\doteq$ to denote this type of agreement to all digits shown, as in $\pi \doteq 3.1415$.

When intervals of nearby numbers occur in the book, we use the notation 1.2345_{67}^{89} to denote $[1.234567, 1.234589]$.

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As long as a method of solution has been used by several contestants or can be found in the existing literature, we will refrain from giving credit to individuals. We do not suggest that any specific idea originates with us, even though there are many such ideas to be found in the book. Although each chapter bears the name of the author who actually wrote it, in every case there have been substantial contributions from the other authors.

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