Foreword

Everyone loves a contest. I still recall when, in January/February 2002, I first read of the SIAM 100-Digit Challenge in *SIAM News*. Nick Trefethen's short article[1] introduced the 10 problems, then closed with the comment, "Hint: They're hard! If anyone gets 50 digits in total, I will be impressed." To an incorrigible computational mathematician like me, that was one giant red flag, an irresistible temptation. As it turned out I did submit an entry, in partnership with two other colleagues, but we failed to get the correct answer on at least 1 of the 10 problems and thus did not receive any award. But it was still a very engaging and enjoyable exercise.

This book shows in detail how each of these problems can be solved, as described by four authors who, unlike myself, belonged to winning teams who successfully solved all 10 problems. Even better, the book presents multiple approaches to the solution for each problem, including schemes that can be scaled to provide thousand-digit accuracy if required and can solve even larger related problems. In the process, the authors visit just about every major technique of modern numerical analysis: matrix computation, numerical quadrature, limit extrapolation, error control, interval arithmetic, contour integration, iterative linear methods, global optimization, high-precision arithmetic, evolutionary algorithms, eigenvalue methods, and many more (the list goes on and on).

The resulting work is destined to be a classic of modern computational science—a gourmet feast in 10 courses. More generally, this book provides a compelling answer to the question, "What is numerical analysis?" In this book we see that numerical analysis is much more than a collection of Victorian maxims on why we need to be careful about numerical round-off error. We instead see first hand how the field encompasses a large and growing body of clever algorithms and mathematical machinery devoted to efficient computation. As Nick Trefethen once observed [Tre98], "If rounding errors vanished, 95% of numerical analysis would remain."

As noted above, the authors of this book describe techniques that in many cases can be extended to compute numerical answers to the 10 problems to an accuracy of thousands of digits. Some may question why anyone would care about such prodigious precision, when in the "real" physical world, hardly any quantities are known to an accuracy beyond about 12 decimal digits. For instance, a value of π correct to 20 decimal digits would suffice to calculate the circumference of a circle around the sun at the orbit of the earth to within the width of an atom. So why should anyone care about finding any answers to 10,000-digit accuracy?

[1] See p. 1 for the full text.

In fact, recent work in experimental mathematics has provided an important venue where numerical results are needed to very high numerical precision, in some cases to thousands of decimal digits. In particular, precision on this scale is often required when applying integer relation algorithms[2] to discover new mathematical identities. An integer relation algorithm is an algorithm that, given n real numbers $(x_i,\ 1 \leqslant i \leqslant n)$, in the form of high-precision floating-point numerical values, produces n integers, not all zero, such that $a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0$.

The best-known example of this sort is a new formula for π that was discovered in 1995:

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6} \right).$$

This formula was found by a computer program implementing the PSLQ integer relation algorithm, using (in this case) a numerical precision of approximately 200 digits. This computation also required, as an input real vector, more than 25 mathematical constants, each computed to 200-digit accuracy. The mathematical significance of this particular formula is that it permits one to directly calculate binary or hexadecimal digits of π beginning at any arbitrary position, using an algorithm that is very simple, it requires almost no memory, and does not require multiple-precision arithmetic [BBP97, AW97, BB04, BBG04]. Since 1996, numerous additional formulas of this type have been found, including several formulas that arise in quantum field theory [Bai00].

It's worth emphasizing that a wide range of algorithms from numerical analysis come into play in experimental mathematics, including more than a few of the algorithms described in this book. Numerical quadrature (i.e., numerical evaluation of definite integrals), series evaluation, and limit evaluation, each performed to very high precision, are particularly important. These algorithms require multiple-precision arithmetic, of course, but often also involve significant symbolic manipulation and considerable mathematical cleverness as well.

In short, most, if not all, of the techniques described in this book have applications far beyond the classical disciplines of applied mathematics that have long been the mainstay of numerical analysis. They are well worth learning, and in many cases, rather fun to work with. Savor every bite of this 10-course feast.

David H. Bailey
Chief Technologist
Computational Research Department
Lawrence Berkeley National Laboratory, USA

[2]Such algorithms were ranked among *The Top 10*—assembled by Jack Dongarra and Francis Sullivan [DS00]—of algorithms "with the greatest influence on the development and practice of science and engineering in the 20th century."