Note on Chapter 3

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On setting $R_{jl}(k) = 4/D(k)$ (first display on p. 56), the partial-fraction decomposition of $R_{jl}(k)$ is given by

$$R_{jl}(k) = \sum_{\nu=1}^{4} \frac{4}{D'(k_\nu)} \frac{1}{k - k_\nu},$$

provided the poles k_ν, $\nu = 1, 2, 3, 4$, are distinct. It is easily verified that R_{jl} has multiple poles if $j = l$, or if j and l belong to the sequence $s_q = (q^2 - q + 2)/2$, $q = 0, 1, 2, \ldots$, with an offset of 2, that is $j = s_{q_1}$ and $l = s_{q_2}$ with $|q_1 - q_2| = 2$. Otherwise, if there are no multiple poles, it follows that

$$g_{jl} = \sum_{k=1}^{\infty} R_{jl}(k) = -\sum_{\nu=1}^{4} \frac{4}{D'(k_\nu)} \psi(1 - k_\nu). \tag{1}$$

The expression (1) has been used to determine g_{24}, g_{17}, g_{13}, g_{35}. On simplification by means of the functional equations (3) and the values $\psi(1) = -\gamma$, $\psi'(1) = \pi^2/6$, the Maple results at the bottom of p. 56 are precisely recovered.
Consider next the case $j = l$, in which $k_1 = k_3$, $k_2 = k_4$. The partial-fraction decomposition of $R_{jj}(k)$ is now given by

$$R_{jj}(k) = \frac{4}{(k - k_1)^2 (k - k_2)^2} = \frac{4}{(k_1 - k_2)^2} \left(\frac{1}{k - k_1} - \frac{1}{k - k_2} \right)^2 =$$
$$= \frac{4}{(k_1 - k_2)^2} \left(\frac{1}{(k - k_1)^2} + \frac{1}{(k - k_2)^2} \right) - \frac{8}{(k_1 - k_2)^3} \left(\frac{1}{k - k_1} - \frac{1}{k - k_2} \right).$$

Then it follows that

$$g_{jj} = \sum_{k=1}^{\infty} R_{jj}(k)$$
$$= \frac{4}{8j - 7} [\psi'(1 - k_1) + \psi'(1 - k_2)] + \frac{8}{(8j - 7)^{3/2}} [\psi(1 - k_1) - \psi(1 - k_2)]. \tag{2}$$

The expression (2) has been used to determine g_{11}, g_{22}, g_{33}, and on simplification there is agreement with the Maple results on p. 57.
The Maple result for g_{33} can be further simplified by use of the following properties of the ψ function:

- functional equations [AS84, form. 6.3.5]

$$\psi(z + 1) = \frac{1}{z} + \psi(z), \quad \psi'(z + 1) = -\frac{1}{z^2} + \psi'(z); \tag{3}$$

- reflection formulas [AS84, form. 6.3.7]

$$\psi(1/2+z)-\psi(1/2-z) = \pi\tan(\pi z),\ \psi'(1/2+z)+\psi'(1/2-z) = \pi^2\sec^2(\pi z). \tag{4}$$

By repeated use of (3) we establish that

$$\psi\left(\frac{7\pm\sqrt{17}}{2}\right) = \sum_{m=0}^{2}\frac{2}{2m+1\pm\sqrt{17}} + \psi\left(\frac{1\pm\sqrt{17}}{2}\right),$$

$$\psi'\left(\frac{7\pm\sqrt{17}}{2}\right) = -\sum_{m=0}^{2}\frac{4}{(2m+1\pm\sqrt{17})^2} + \psi'\left(\frac{1\pm\sqrt{17}}{2}\right).$$

Next it follows that

$$\begin{aligned}
&\psi\left(\frac{7-\sqrt{17}}{2}\right) - \psi\left(\frac{7+\sqrt{17}}{2}\right) = \\
&= \sum_{m=0}^{2}\frac{4\sqrt{17}}{(2m+1)^2-17} + \psi\left(\frac{1-\sqrt{17}}{2}\right) - \psi\left(\frac{1+\sqrt{17}}{2}\right) \\
&= -\frac{1}{4}\sqrt{17} - \pi\tan(\sqrt{17}\,\pi/2), \\
&\psi'\left(\frac{7-\sqrt{17}}{2}\right) + \psi'\left(\frac{7+\sqrt{17}}{2}\right) = \\
&= -\sum_{m=0}^{2}\frac{8[(2m+1)^2+17]}{[(2m+1)^2-17]^2} + \psi'\left(\frac{1-\sqrt{17}}{2}\right) + \psi'\left(\frac{1+\sqrt{17}}{2}\right) \\
&= -\frac{145}{16} + \pi^2\sec^2(\sqrt{17}\,\pi/2),
\end{aligned}$$

by means of the reflection formulas (4).
Insert these results into the Maple expression for g_{33} on top of p. 57. Then we obtain

$$\begin{aligned}
g_{33} &= \frac{4}{17}\left(-\frac{145}{16} + \pi^2\sec^2(\sqrt{17}\,\pi/2)\right) + \frac{8\sqrt{17}}{289}\left(-\frac{1}{4}\sqrt{17} - \pi\tan(\sqrt{17}\,\pi/2)\right) \\
&= -\frac{9}{4} + \frac{4\pi}{289}\sec^2(\sqrt{17}\,\pi/2)\,[17\pi - \sqrt{17}\,\sin(\sqrt{17}\,\pi)],
\end{aligned} \tag{5}$$

in accordance with the *Mathematica* expression (middle of p. 57).

In the same manner we derive an expression for g_{jj}, as given by (2), in terms of elementary functions. The idea is to reduce

$$\psi(1-k_{1,2}) = \psi\left(\frac{2j+1\mp\sqrt{8j-7}}{2}\right),\ \psi'(1-k_{1,2}) = \psi'\left(\frac{2j+1\mp\sqrt{8j-7}}{2}\right),$$

to $\psi((1 \mp \sqrt{8j-7})/2)$, $\psi'((1 \mp \sqrt{8j-7})/2)$ by repeated use of (3). Next one should use the reflection formulas (4). Omitting the details we present the final result

$$g_{jj} = -4 \sum_{m=0}^{j-1} \frac{1}{(m^2+m+2-2j)^2} + \\ + \frac{4\pi}{(8j-7)^2} \sec^2(\sqrt{8j-7}\,\pi/2)\,[(8j-7)\pi - \sqrt{8j-7}\ \sin(\sqrt{8j-7}\,\pi)]. \quad (6)$$

The latter expression is valid if $\sqrt{8j-7} \neq 2q+1$, (odd integer), $q = 0, 1, 2, \ldots$, or equivalently, if $j \neq (q^2+q+2)/2$. For such values of j, the $(m = q)$-term of the sum in (6) and $\sec^2(\sqrt{8j-7}\,\pi/2)$ become singular; of course, these singularities cancel. In the case $j = (q^2+q+2)/2$ for some $q = 0, 1, 2, \ldots$, one has $\sqrt{8j-7} = 2q+1$, $k_1 = 1 - j + q$, $k_2 = -j - q$, whereupon the expression (2) simplifies to

$$g_{jj} = \frac{4}{(2q+1)^2}\,[\psi'(j-q) + \psi'(1+j+q)] + \frac{8}{(2q+1)^3}\,[\psi(j-q) - \psi(1+j+q)].$$

By means of (3) and the known value $\psi'(1) = \pi^2/6$ the latter expression can be further reduced to

$$g_{jj} = \frac{4\pi^2}{3(2q+1)^2} - \frac{4}{(2q+1)^2}\left[\sum_{m=1}^{j-q-1} \frac{1}{m^2} + \sum_{m=1}^{j+q} \frac{1}{m^2}\right] - \frac{8}{(2q+1)^3} \sum_{m=j-q}^{j+q} \frac{1}{m}, \quad (7)$$

valid for $j = (q^2+q+2)/2$ for some $q = 0, 1, 2, \ldots$. In the special cases $q = 0$, $j = 1$, and $q = 1$, $j = 2$, the expression (7) leads again to the Maple results for g_{11}, g_{22} on p. 56. As an additional result for $q = 2$, $j = 4$, one has

$$g_{44} = \frac{4\pi^2}{75} - \frac{11057}{22500}.$$