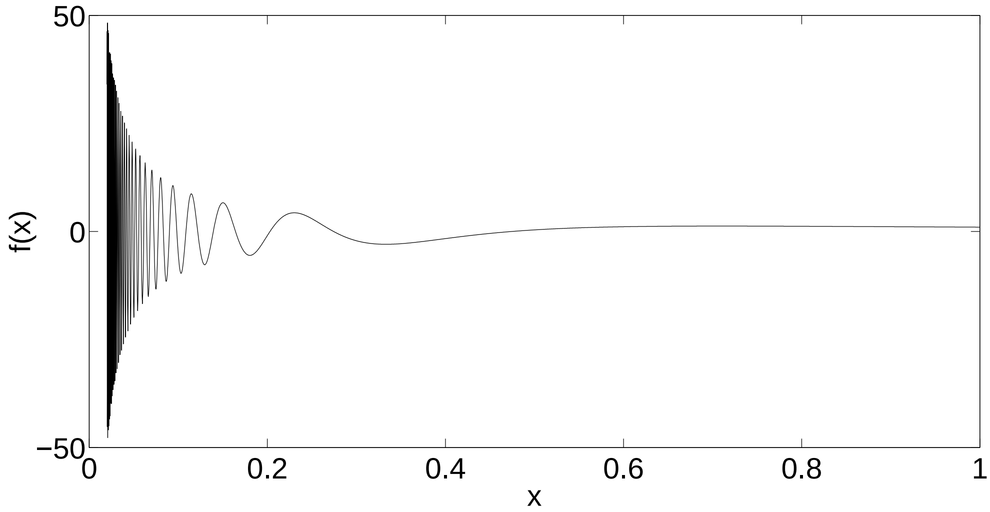
Nick Trefethen's 100 digit challenge

- Solve 10 problems with an accuracy of 10 digits each.
  Hence maximum of 100 correct digits.
- 20 teams with 100 correct digits.
  5 teams with 99 correct digits.
  Altogether 94 teams from 25 countries.
- Chemnitz University teams:
  Thomas Grund
  Gerd Kunert, Ulf Kähler (problem 2)
  Analytical help for problem 10 from Wunderlich/Starkloff.
- Software:
  Matlab (double precision)
  Maple (arbitrary precision)
  gmp (GNU multiprecision library with C), cln (C++ library)
  Other teams used Mathematica (superior over Maple?!)
- `web.comlab.ox.ac.uk/oucl/work/nick.trefethen/hundred.html`

Problem 1

Problem: What is $\lim_{\varepsilon\to 0}\int_{\varepsilon}^{1} x^{-1}\cos(x^{-1}\ln x)\ dx$?
Answer: 0.32336 74316 77
Solution:

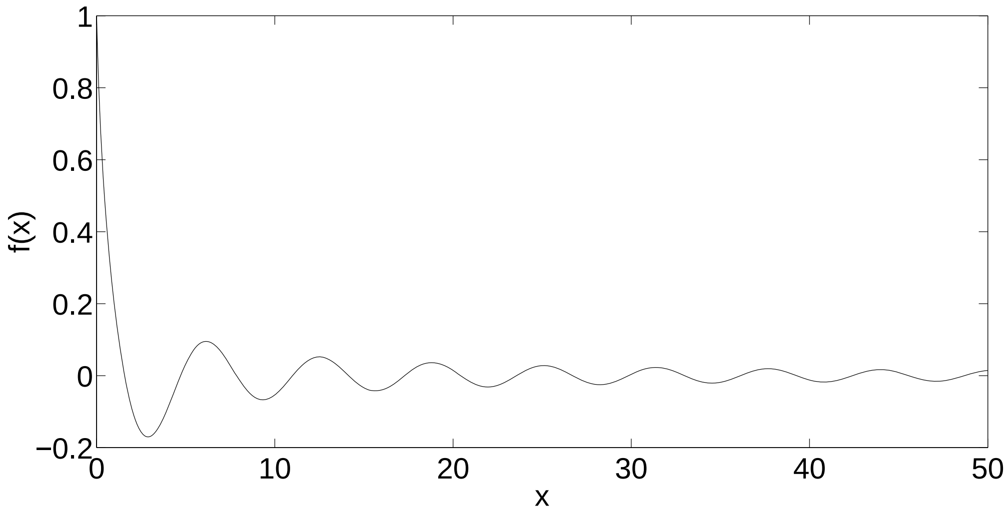
- function $f(x) = x^{-1}\cos(x^{-1}\ln x)$



- $$\int_0^1 x^{-1}\cos(x^{-1}\ln x)\ dx = \int_0^{\infty} \mathcal{W}'(x)\cos(x)\ dx$$

- $\mathcal{W}(x)\exp\mathcal{W}(x) = x, \qquad \mathcal{W}'(x) = \dfrac{\mathcal{W}(x)}{(1+\mathcal{W}(x))x}$

- function $f(x) = \mathcal{W}'(x)\cos(x)$



Problem 1

- split integral $\Rightarrow \displaystyle\sum_{k=0}^{\infty} \int_{2k\pi}^{(2k+2)\pi} \mathcal{W}'(x)\cos(x)\ dx$
- each integral: Gauss-Legendre integration
- $\displaystyle\int_{-1}^{1} f(x)\ dx \approx \sum_{i=0}^{n} w_i f(x_i)$
- x_i zeros of $L_{n+1}(x)$, $w_i = \dfrac{2}{(1-x_i^2)(L'_{n+1}(x_i))^2}$
- L_n is the n-th Legendre polynomial
- but in praxis: just a few lines of code
- Quarteroni, Sacco, Saleri: Numerical Mathematics !!!
- $100,000$ terms of the sum give 10 digits
- other solutions: integrating by parts
- either $\displaystyle\int_{1}^{\infty} \frac{\sin(x\ln x)}{u^n} f_n(x)\ dx - c_n$ or $\displaystyle\int_{0}^{\infty} \mathcal{W}^{(4)} \sin x\ dx - c$

Problem 2

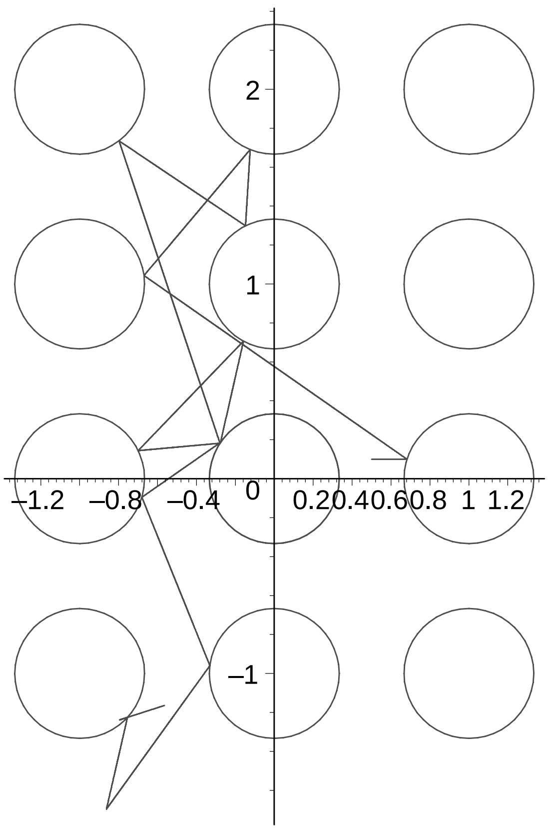
Problem: A photon moving at speed 1 in the xy plane starts at $t = 0$ at $(x, y) = (0.5, 0.1)$ heading due east. Around every integer lattice point (i, j) in the plane, a circular mirror of radius 1/3 has been erected. How far from the origin is the photon at $t = 10$?

Answer: 0.99526 29194 43354

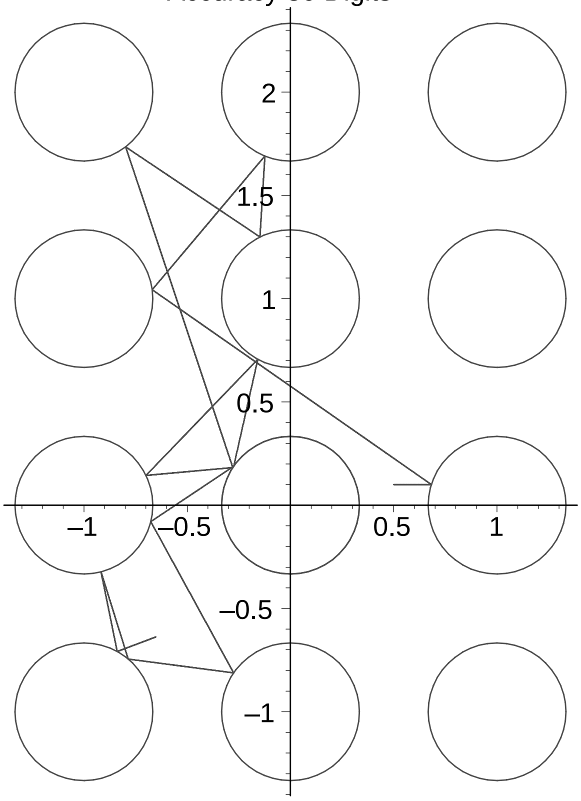
Solution:

- Standard geometry and trigonometry. Formulas can be written without using trigonometric functions.
- Loss of accuracy of about 15 digits, i.e. compute with 25 digits to achieve a final accuracy of 10 digits.
  Hence high precision arithmetic (e.g. Maple) is mandatory.
- Comparison of different accuracy:

Accuracy 10 Digits



Accuracy 30 Digits



Problem 3

Problem: The infinite matrix A with entries $a_{11} = 1$, $a_{12} = 1/2$, $a_{21} = 1/3$, $a_{13} = 1/4$, $a_{22} = 1/5$, $a_{31} = 1/6$, etc., is a bounded operator on l^2. What is $\|A\|$?
Answer: 1.2742 24152 8212
Solution:

- $A = \begin{pmatrix} 1 & 1/2 & 1/4 & \dots \\ 1/3 & 1/5 & & \\ 1/6 & & \ddots & \\ \vdots & & & \end{pmatrix}, \qquad A_{ij} = \frac{1}{(i+j-1)(i+j-2)/2+j}$
- $A_n \in \mathbb{R}^{n \times n}$ upper-left part of A, assume $\|A\| = \lim_{n\to\infty} \|A_n\|$
- $\|A_n\|$ is the square root of the largest eigenvalue of $A_n^T A_n$
- apply the power method to calculate this largest eigenvalue
- power method: $z^{(k)} = Av^{(k-1)}$, $v^{(k)} = z^{(k)}/\|z^{(k)}\|$
- convergence rate depends on the ratio of the two largest eigenvalues
- here: convergence within 5 iterations
- n up to 10000
- other solutions: extrapolation gives up to ≈ 16 digits
- Rolf Strebel: 40 digits? How?

Problem 4

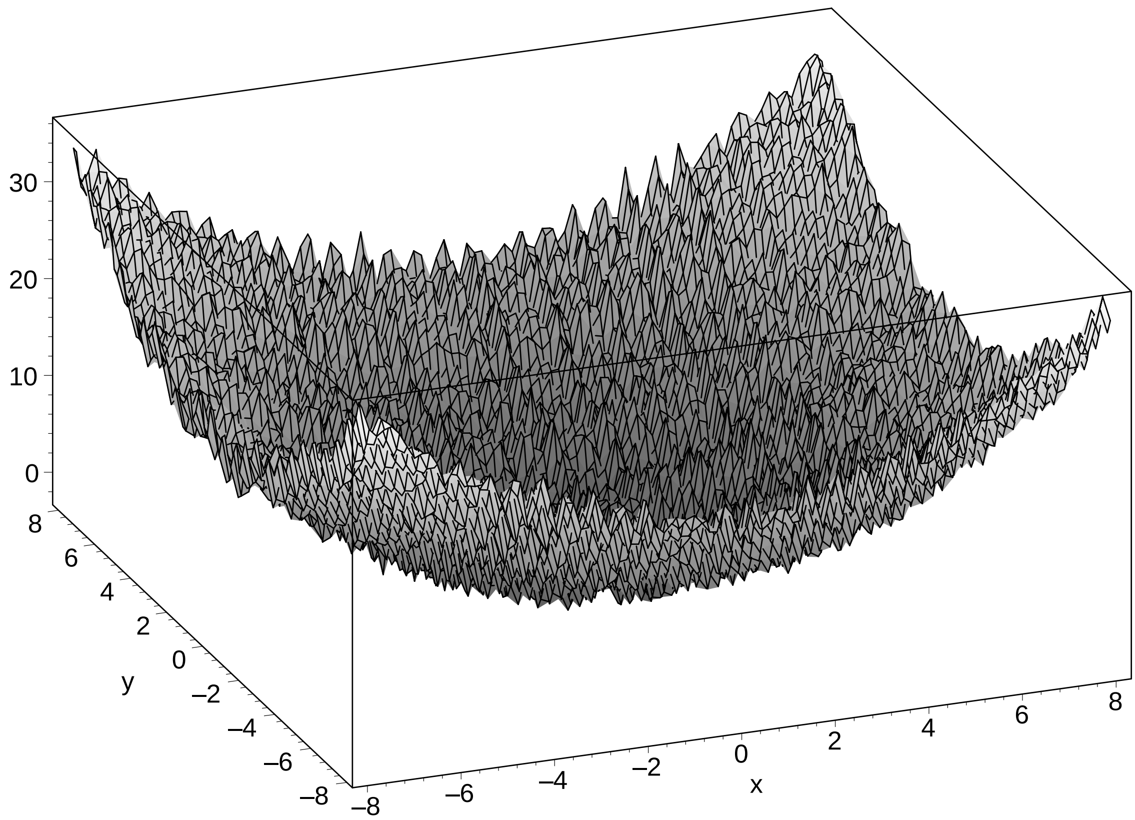
Problem: What is the global minimum of the function

$$f(x,y) := \exp(\sin(50x)) + \sin(60\exp(y)) + \sin(70\sin(x)) + \sin(\sin(80y)) - \sin(10(x+y)) + (x^2+y^2)/4 \text{ ?}$$

Answer: -3.30686 86474 75

Solution:

- Global behaviour is quadratic
  Local behaviour: highly oscillating.

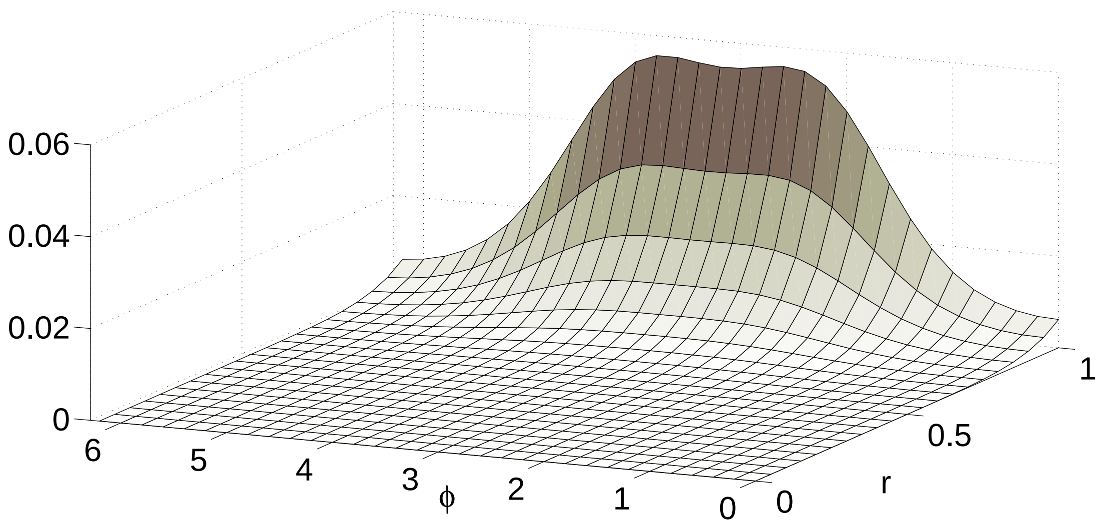


- Minimum somewhere in $[-5,5]^2$.
- Brute force search for minimum point on equidistributed 2D mesh with different mesh sizes.
  Start with $[-5,5]^2$ and gradually make the domain smaller.
- Once the spike with the minimum is found, use standard minimization methods (steepest descent, graphical minimization).

Problem 5

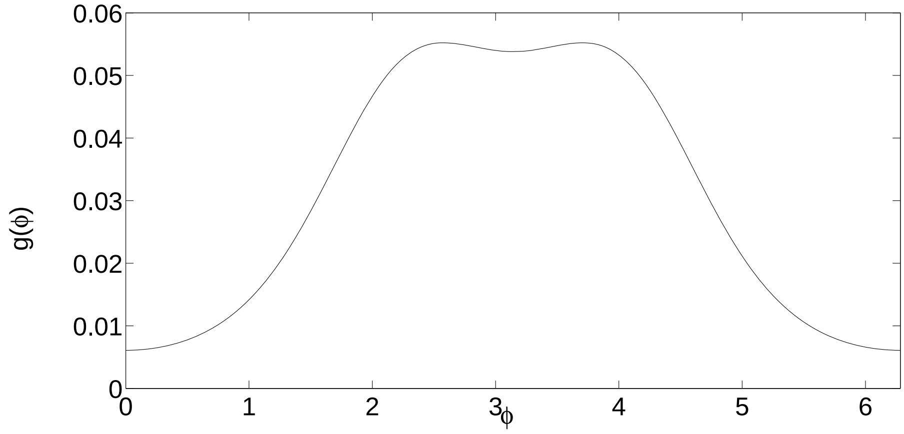
Problem: Let $f(z) = 1/\Gamma(z)$, where $\Gamma(z)$ is the gamma function, and let $p(z)$ be the cubic polynomial that best approximates $f(z)$ on the unit disk in the supremum norm $\|\cdot\|_\infty$. What is $\|f-p\|_\infty$?
Answer: 0.21433 52345 90459 6
Solution:

- $p(z) = az^3 + bz^2 + cz + d, \quad z = r\mathrm{e}^{i\phi}, \quad a, b, c, d \in \mathbb{C}$
- idea: calculate the points z_i where the maximum norm of $p(z) - f(z)$ is attained
- try to lower the value of $|p(z) - f(z)|^2$ at these points
- the points z_i change as well and also more and more points have to be considered
- start by calculating the polynomial $p_2(z)$ that best approximates $f(z)$ in the L^2 norm $\Rightarrow$ system of four linear equations with approximate solution $a \approx 0.655, b \approx 0.577, c \approx 1, d \approx 0.00$
- for this values $|f-p|^2$ in the (r, ϕ) plane looks like



Problem 5

- assume: maximum is attained on the boundary $\Rightarrow \; z_i = \mathrm{e}^{i\phi_i}$
- define $g(\phi) = |f(\mathrm{e}^{i\phi}) - p(\mathrm{e}^{i\phi})|^2$
- g for the initial values of a, b, c, d looks like



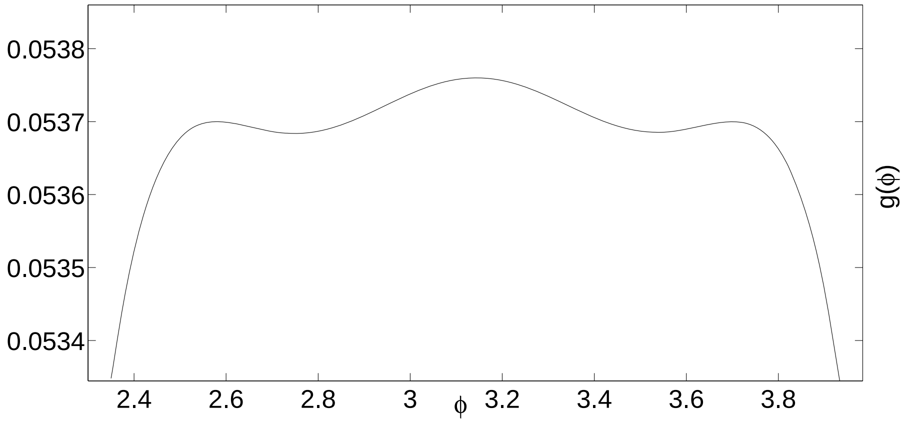
- maximum of g is ≈ 0.0552; choose $\phi_1 = 2.58, \phi_2 = 3.7, n = 0.0552$
- consider the system of equations

$$
\begin{aligned}
g(\phi_i) - n &= 0 \qquad i = 1, 2(,3,4,5) \\
g'(\phi_i) &= 0 \qquad i = 1, 2(,3,4,5)
\end{aligned}
$$

- solve this system with the unknowns $a, b, c, d \in \mathbb{C}, \phi_1, \phi_2 \in \mathbb{R}$ with the Newton method for smaller and smaller values of n

Problem 5

- first case: it is not possible to solve the system anymore $\Rightarrow n$ was chosen to small
- second case: a new hump occurs:



- introduce a new variable ϕ_3 with initial value π
- by repeating this process one ends up with five humps meaning five variables ϕ_i
- the square root of the final value of n is the solution
- other solutions: Maximum Modulus Theorem: supremum occurs on the circumference of the disk; coefficients of the polynomial must be real; Tonelli: supremum is attained at at least 5 points
- Remez algorithm?
- Standard minimization of $\|f - p(a, b, c, d)\|_\infty$ with respect to a, b, c, d is very ill-conditioned because of maximum norm. Descent algorithms almost fail since $\|\cdot\|_\infty$ is not differentiable w.r.t. a, b, c, d.

Problem 6

Problem: A flea starts at $(0,0)$ on the infinite 2D integer lattice and executes a biased random walk: At each step it hops north or south with probability 1/4, east with probability $1/4+\varepsilon$, and west with probability $1/4-\varepsilon$. The probability that the flea returns to $(0,0)$ sometime during its wanderings is 1/2. What is ε ?
Answer: 0.06191 39544 7400
Solution:

- Main difficulty: Find a suitable model (start with 1D analogue).
- Let $P_{i,k}^n$ be the probability that the flea has reached point (i,k) within $2n$ steps without returning to $(0,0)$.
- Compute $P_{i,k}^n$ iteratively from P_*^{n-1}.

  This involves only the transition probabilities and the values of P_*^{n-1} from neighbouring points.
- Store all P_*^n in a $(2n+1)\times(2n+1)$ matrix.
  Matrix can be overwritten when calculating P_*^{n+1}.
- Compute $P_*^{100}, P_*^{200} \dots P_*^{1600}$ and extrapolate P_*^{∞}.
  Use bisection to find the transition probability ε.
- Verification:
  - $\varepsilon=0$ gives return probability 1.
  - North/south transition probability 0 produces 1D model.

Problem 6

Analytical solution: (proposed by F. Bornemann)

- Let return probability be p_ε.
  Then expected number of visits to the origin (including start) is $m_\varepsilon = 1/(1-p_\varepsilon)$.
- Define characteristic function $\phi_\varepsilon(\theta)$ of the random walk.
- Compute m_ε explicitly:

$$m_\varepsilon = \frac{2\sqrt{2}K\left(1-\frac{1+8\varepsilon^2-\sqrt{1-16\varepsilon^2}}{1+8\varepsilon^2+\sqrt{1-16\varepsilon^2}}\right)}{\pi\sqrt{1+8\varepsilon^2+\sqrt{1-16\varepsilon^2}}}$$

$$= \frac{\sqrt{2}}{\mathsf{AGM}(1+8\varepsilon^2-\sqrt{1-16\varepsilon^2}, 1+8\varepsilon^2+\sqrt{1-16\varepsilon^2})}$$

  K: complete elliptic integral of first kind
  AGM: arithmetic geometric mean

- $p_\varepsilon = 1/2$ means $m_\varepsilon = 2$. Hence solve by bisection

$$\mathsf{AGM}(1+8\varepsilon^2-\sqrt{1-16\varepsilon^2}, 1+8\varepsilon^2+\sqrt{1-16\varepsilon^2}) = 1/\sqrt{2}$$

- Literature: F. Spitzer, Principles of random walk, Springer 1976

Problem 7

Problem: Let A be the $20,000 \times 20,000$ matrix whose entries are zero everywhere except for the primes $2, 3, 5, 7, \ldots, 224737$ along the main diagonal and the number 1 in all the positions a_{ij} with $|i - j| = 2, 4, 8, \ldots, 16384$. What is the $(1, 1)$ entry of A^{-1}?
Answer: 0.72507 83462 68401 16746 86877 19251 16096 88691 8059
Solution:

- A^{-1}_{11} is the first element of the solution vector x that solves $Ax = e_1$ where $e_1 = (1, 0, 0, \ldots, 0)^T$.
- solution of a sparse linear system: conjugated gradient method
- number of iterations: $400 (2049 \times 2049)$, $582 (4097 \times 4097)$, $\ldots$
- preconditioning the diagonal: 17 iterations for the matrix A

Problem 8

Problem: A square plate $[-1,1]^2$ is at temperature $u = 0$. At time $t = 0$ the temperature is increased to $u = 5$ along one of the four sides while being held at $u = 0$ along the other three sides, and heat then flows into the plate according to $u_t = \Delta u$. When does the temperature reach $u = 1$ at the center of the plate?

Answer: 0.42401 13870 33688

Solution:

- Homogenize the boundary conditions:

$$u(x, y, t) = u_\Gamma(x, y) + u_0(x, y, t).$$

- Find $u_\Gamma(x, y)$ that satisfies the b.c. and $\Delta u_\Gamma = 0$.
  Fourier ansatz for $\Delta u_\Gamma = 0$ and $u_\Gamma = 0$ on $x = 1$ or $y = \pm 1$:

$$u_\Gamma(x, y) = \sum_{k=1}^{\infty} a_k \sinh(k\pi(x-1)/2) \sin(k\pi(y-1)/2)$$

  Boundary conditions $u_\Gamma = 5$ on $x = -1$ give

$$a_k = \frac{10}{k\pi \sinh(k\pi)}(1 - (-1)^k)$$

Problem 8

- Find u_0 which satisfies

$$\begin{aligned} u_{0,t} - \Delta u_0 &= 0 \\ u_0(\Gamma, t) &= 0 \\ u_0(x, y, 0) &= -u_\Gamma(x, y) \end{aligned}$$

- Standard Fourier ansatz (for first two equations):

$$u_0 = \sum_{k,l=1}^{\infty} a_{k,l}\, e^{-(\lambda_k^2+\lambda_l^2)t} \sin(\lambda_k(x-1)) \sin(\lambda_l(y-1))$$

with $\lambda_k = k\pi/2$.

Fourier ansatz means that

$$\left\{ \sin(\lambda_k(x-1)) \sin(\lambda_l(y-1)) \right\}_{k,l=1}^{\infty}$$

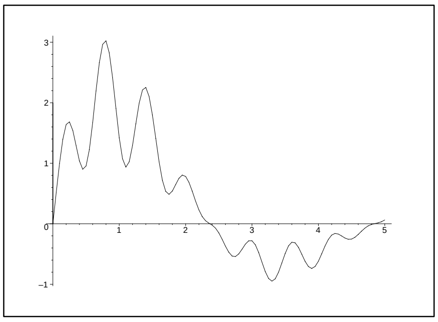
forms an orthogonal system on $L_2([-1,1]^2)$. This gives easily

$$a_{k,l} = \frac{20k(-1)^k}{\pi^2 l(k^2+l^2)}(1-(-1)^l)$$

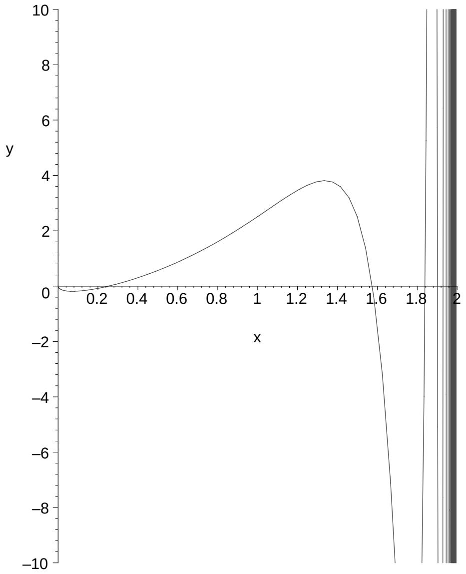
- Assemble the solution and solve for $u(0,0,t) = 1$. The solution is about $t \approx 0.4$. Then the double sum converges very rapidly (because of the exponential factor).
Simple bisection provides the exact t.

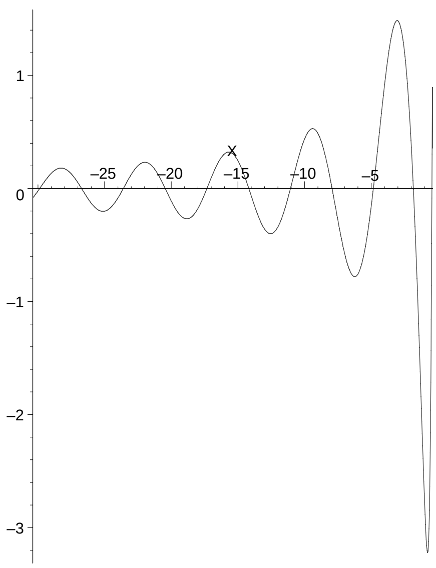
Problem:
The integral $I(\alpha) = \int_0^2 (2+\sin(10\alpha))x^\alpha \sin(\alpha/(2-x))\,dx$ depends on the parameter α. What is the value $\alpha \in [0,5]$ at which $I(\alpha)$ achieves its maximum?
Answer: 0.78593 36743 5047
Solution:

- function $I(\alpha) \Rightarrow\ \alpha$ between 0.7 and 0.9.



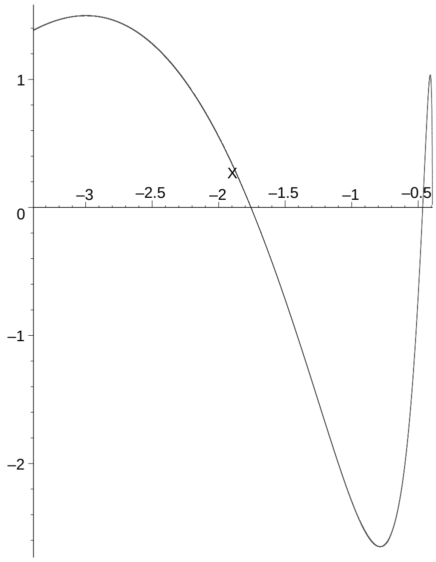
- $f(x) = (2+\sin(10\alpha))x^\alpha \sin(\alpha/(2-x))$
- necessary condition: $\int_0^2 \partial f/\partial\alpha\ dx = 0$
- $\partial f/\partial\alpha$ for α near the solution an after transformation $u = \alpha/(x-2)$





Problem 9

- detail



- integral: fine discretization on the right, other part with Gauss-Legendre
- bisection to get the value for α
- other solutions: the integral $I(\alpha)$ has a closed form expression in terms of Meijer's G-function

Problem 10

Problem: A particle at the center of a 10×1 rectangle undergoes Brownian motion (i.e., 2D random walk with infinitesimal step lengths) till it hits the boundary. What is the probability that it hits at one of the ends rather than at one of the sides?
Answer: 0.38375 87979 25e-6
Solution:

- Analytic approach by Wunderlich/Starkloff:
  Assume rectangle with side lengths $2a \geq 2b$.

$$F_a(x) := 1 - \frac{1}{\sqrt{2\pi x}} \sum_{k=-\infty}^{\infty} (-1)^k \int_{-a}^{a} \exp\left(-\frac{(z-2ka)^2}{2x}\right) \mathrm{d}z$$

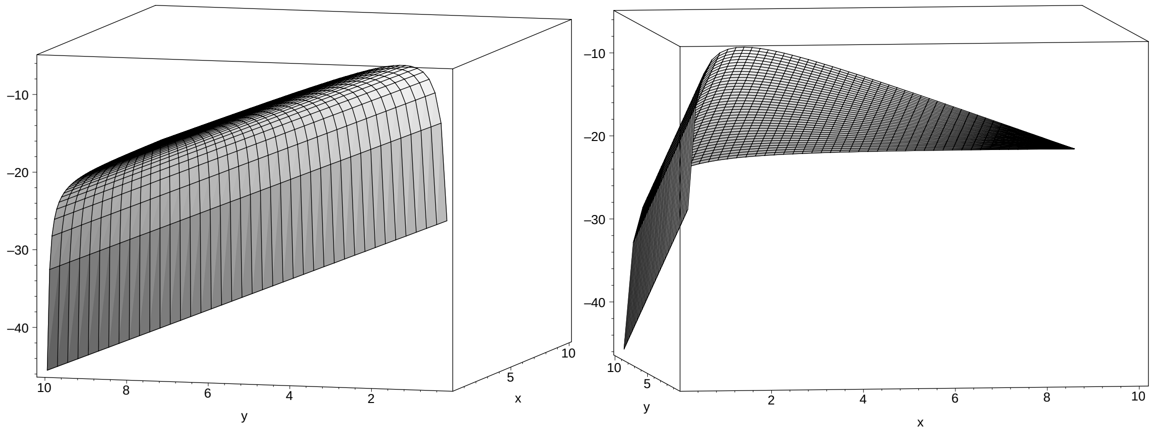
$$f_a(x) := \frac{\partial F_a(x)}{\partial x} \qquad f_b(y) := \frac{\partial F_b(y)}{\partial y}$$

$$P = \iint_{0 \leq x \leq y} f_a(x)\, f_b(y)\, \mathrm{d}x\, \mathrm{d}y$$

- Difficulty: How and where to start?
  First try: Approximate infinite sums by finite sums. Trial and error to get sufficient accuracy:
  Sum in f_a from about $k = -10 \ldots 10$.
  Sum in f_b from about $k = -40 \ldots 40$
  (both depending mildly on x, y).
  Use high precision arithmetic (e.g. Maple).

Problem 10

- Visualize $\log(f_a(x)\, f_b(y))$ (2 views, log plot):



  Integration up to $y = 12$ should be sufficient for 10 accurate digits.

- Numerical integration over triangular domain $0 \leq x \leq y \leq 12$.
  - Maple fails. Alternatives:
  - Split into rectangles (two 1D integrals by Maple) and small triangular domains (multilevel-like numerical integration).
  - Simplify 2D integral to 1D integral:

$$P = \int_{y=0}^{\infty} f_b(y)(F_a(y) - F_a(0))\,\mathsf{d}y$$

Problem 10

Advanced solution by F. Bornemann:

- Let $p(x)$ be the probability that a particle starting at the point x of the rectangle R hits Γ_a rather than Γ_b.

  Then p is the solution of the elliptic boundary value problem on R:

$$\Delta p = 0 \qquad p|_{\Gamma_a} = 1 \qquad p|_{\Gamma_b} = 0.$$

- Fourier series for centre:

$$p(0,0) = \sum_{k=0}^{\infty} \frac{4(-1)^k}{\pi(2k+1)\cosh\left(\frac{(2k+1)\pi}{2}\frac{a}{b}\right)}$$

  For $a/b = 10$ very rapid convergence: First term gives 14 accurate digits!

- Even closed expression possible:

$$p(0,0) = \frac{2}{\pi} \arcsin \frac{\sqrt{2} - (\sqrt{5}-2)(3+2\sqrt[4]{5})}{\sqrt{2} + (\sqrt{5}-2)(3+2\sqrt[4]{5})}$$

- S. Karlin: A first course in stochastic processes, Academic Press, 1966.

Alternatives:

- Brownian motion can be describes by parabolic PDE with point source at $x = (0,0), t = 0$.
  Again no numerical integration necessary.