100 Digits Solutions
University of Delaware Mathematical Sciences

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(thanks to Peter Monk and Y. J. Leung)

Our team consisted of two faculty numerical analysts, two grad students in discrete math, an undergrad in math, and a grad student in CS. Everyone in the core group was an active and useful participant—reinforcing the lore that numerical analysts are minority contributors to their own field!

According to the NA-Digest checksum, our score is $\leq$ 98. Although we are concerned that this will not even put us in the top ten (the willingness of academics to waste time was probably underestimated), we can no longer identify improvements.

We enjoyed the evolution from ugly brute force to (in most cases) an elegant, efficient approach and then to (in some cases) an appreciation of more universal applicability of some new ideas. All of us came away feeling that we had added to our own personal bags of tricks.

Somewhat surprisingly to some of us old-timers, many of the problems were at least partially defanged when implemented in Maple. We are happy that (in the end) we were not led to throw obscene hardware resources at the list. Wetware was the star of our show, with software in an important supporting role and hardware a mere bit player.

Problems that we definitely feel we got "right" were numbers 6, 7, 8, and 10; we also like our answers to numbers 1 and 4 but allow that still slicker solutions might exist. Number 5 was disappointing in that the most elegant avenue (LNT's CF method) didn't really work out for us, and the result presented is perhaps our weakest link. Numbers 3 and 9 did not use any especially clever tricks or arguments, though they were not exactly easy. Number 2 was the most puzzling psychologically: Given variable precision, the answer was completely straightforward. Did we miss a cute trick?

1. 0.3233674317
   We used a substitution to transform to $\int_0^\infty \cos(u)W'(u)\,du$, where W is Lambert's W function. Integrate from 0 to $2k\pi$ directly (using Maple numerics, for instance; $k = 16$ was found to be effective). Then we integrated by parts several times, putting more derivatives on W so that it decays more rapidly. Eventually the remaining integral becomes easy to evaluate or negligible. The only contributing boundary terms are at $2k\pi$.

2. 0.9952629194
   This was simulated directly in Maple using variable precision arithmetic. The solution was run forwards and backwards to ensure sufficient agreement.

3. 0.1274224153×10^1
   The matrix was truncated to various $N \times N$ principal minors. The norms so computed were extrapolated (in one case, by rational interpolation, and in another, by a Riemann zeta technique) to arrive at the digits shown. (If one is patient one can simply choose $N \approx 2500$.)

4. $-0.3306868647 \times 10^1$
   It's clear from a detailed enough plot that the function goes below -3.3, so by virtue of the last term the minimizer lies within a distance $\sqrt{4(-3.3-(e^{-1}-4))} \approx 1.16$ of the origin. Without the terms that depend exclusively on x, there is no way to go below -3, so these three terms must collectively get below -0.3. Similarly, the three terms depending on y alone must get below about -1.66. Plotting each case, we find 5 allowable small intervals for each variable. This gives 25 small square regions to check, and f varies slowly on each.

5. 0.2143352346
   Lawson's algorithm (weighted least squares on the circle) was used to get a rough approximation to p. Then MATLAB's `fminsearch` was used to solve the discrete min-max problem iteratively, with more points added near those where the current best error curve was largest. In the end, about 600 adaptively added points on the upper half of the circle seemed to be sufficient.

6. $0.6191395447 \times 10^{-1}$
   The probabilities of visiting the origin starting from each point in the (truncated) lattice form a discrete linear boundary-value problem that can be solved using sparse linear algebra. These can be used to compute the probability of returning to the origin from each of the origin's neighbors. Rootfinding was applied to the computed probability. A 160×160 lattice appeared to be large enough; each probability calculation takes less than 15 seconds.

7. 0.7250783463
   Let $A = D + T$, where D is the diagonal of primes. Then $A^{-1}e_1 = (I + D^{-1}T)D^{-1}e_1 = \frac{1}{2}\left(\sum(-D^{-1}T)^n\right)e_1$. The Neumann series is cheap to evaluate using sparse arithmetic. After about 60 terms Aitken extrapolation was used to get the digits shown here.

8. 0.4240113870
   The steady-state solution was subtracted off to get an IBVP for the transient with homogeneous boundary conditions. The steady-state solution is separable and was expressed exactly as a sinh/sine series. The transient was expressed as a sine series whose coefficients were derived from the steady-state expression. Thanks to the heat kernel, the transient series converges very rapidly at the origin for finite times—eight terms were well more than enough. The steady-state value at the center (by harmonic average) is $5/4$, so rootfinding was used to find the time when the truncated series reaches $-1/4$.

9. 0.7859336744
   Maple was able to find a series for the integral; this failed to give high precision but helped us locate the correct maximum. $I'(\alpha)$ can be expressed (after much experimentation) in a form that seemed to make Maple's numerical integrator happy. Then the secant method was applied to find the root of I'. (An alternative method was to transform I to $(0, \infty)$, then use Gauss–Jacobi quadrature for the left singularity and a combination of asymptotics and Gaussian quadrature for the remainder. This was evaluated at Chebyshev points and the rootfinder was applied to the spectral derivative of this discretization.)

10. $0.3837587979 \times 10^{-6}$
    Brownian motion remains Brownian under conformal mapping, so the problem was mapped to the unit disk with starting point at the origin. There the probability is just the relative arc length of the short sides. The map was constructed using the upper half-plane as an intermediate domain; a rootfinder was used to get the correct elliptic modulus of complete elliptic integrals for the given aspect ratio. Cross-ratio invariance from the half-plane to the disk then gave the images of the rectangle's corners, which in turn gave the desired arc lengths.