these cases. We must weigh the apparent security purchased by requiring predicative definitions against the burden of having to abandon in many cases what we, as mathematicians, consider natural definitions.

2. It is unclear exactly what objects we are committed to when we are committed to Peano Arithmetic. There are plenty of problems in number theory whose proofs use analytic means, for instance. Does commitment to Peano Arithmetic entail commitment to whatever objects are needed for these proofs? More generally, does commitment to a mathematical theory mean commitment to any objects needed for solving problems of that theory? If so, then Gödel's incompleteness theorems suggest that it is open what objects commitment to Peano Arithmetic entails.
3. As Feferman admits, it is unclear how to account predicatively for some mathematics used in currently accepted scientific practice, for instance, in quantum mechanics. In addition, I think that Feferman would not want to make the stronger claim that *all future* scientifically applicable mathematics will be accountable for by predicative means. However, the claim that *currently* scientifically applicable mathematics can be accounted for predicatively seems too time-bound to play an important role in a foundation of mathematics. Though it is impossible to predict all future scientific advances, it is reasonable to aim at a foundation of mathematics that has the potential to support these advances. Whether or not predicativity is such a foundation should be studied critically.
4. Whether the use of impredicative sets, and the uncountable more generally, is needed for ordinary finite mathematics, depends on whether by "ordinary" we mean "current." If so, then this is subject to the same worry I raised for (3). It also depends on where we draw the line on what counts as finite mathematics. If, for instance, Goldbach's conjecture counts as finite mathematics, then we have a statement of finite mathematics for which it is completely open whether it can be proved predicatively or not.

In emphasizing the degree to which concerns about predicativism shape this book, I should not overemphasize it. There is much besides predicativism in this book, as I have tried to indicate. In fact, Feferman advises that we not read his predicativism too strongly. In the preface, he describes his interest in predicativity as concerned with seeing how far in mathematics we can get without resorting to the higher infinite, whose justification he thinks can only be platonic. It may turn out that uncountable sets are needed for doing valuable mathematics, such as solving currently unsolved problems. In that case, Feferman writes, we "should look to see where it is necessary to use them and what we can say about what it is we know when we do use them" (p. ix).

Nevertheless, Feferman's committed anti-platonism is a crucial influence on the book. For mathematics right now, Feferman thinks, "a little bit goes a long way," as one of the essay titles puts it. The full universe of sets admitted by the platonist is unnecessary, he thinks, for doing the mathematics for which we must currently account. Time will tell if future developments will support that view, or whether, like Brouwer's view, it will require the alteration or outright rejection of too much mathematics to be viable. Feferman's book shows that, far from being over, work on the foundations of mathematics is vibrant and continuing, perched deliciously but precariously between mathematics and philosophy.

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The SIAM 100-Digit Challenge: A Study in High-Accuracy Numerical Computing

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SIAM, PHILADELPHIA, PA, USA 2004, XII+306 PP. SOFTCOVER, ISBN 0-89871-561-X, US$57.00

REVIEWED BY JONATHAN M. BORWEIN

Lists, challenges, and competitions have a long and primarily lustrous history in mathematics. This is the story of a recent highly successful challenge. The book under review makes it clear that with the continued advance of computing power and accessibility, the view that "real mathematicians don't compute" has little traction, especially for a newer generation of mathematicians who may readily take advantage of the maturation of computational packages such as *Maple*, *Mathematica*, and MATLAB.

Numerical Analysis Then and Now

George Phillips has accurately called Archimedes the first numerical analyst [2, pp. 165–169]. In the process of obtaining his famous estimate $3 + 10/71 < \pi < 3 + 1/7$, he had to master notions of recursion without computers, interval analysis without zero or positional arithmetic, and trigonometry without any of our modern analytic scaffolding. . . . Two millennia later, the same estimate can be obtained by a computer algebra system [3].

Example 1. A modern computer algebra system can tell one that

$$0 < \int_0^1 \frac{(1-x)^4 x^4}{1+x^2}\,dx = \frac{22}{7} - \pi, \tag{1.1}$$

since the integral may be interpreted as the area under a positive curve.

This leaves us no wiser as to why! If, however, we ask the same system to compute the indefinite integral, we are likely to be told that

$$\int_0^t \cdot = \frac{1}{7}t^7 - \frac{2}{3}t^6 + t^5 - \frac{4}{3}t^3 + 4t - 4\arctan(t).$$

Then (1.1) is now rigorously established by differentiation and an appeal to Newton's Fundamental theorem of calculus. □

While there were many fine arithmeticians over the next 1500 years, this anecdote from Georges Ifrah reminds us that mathematical culture in Europe had not sustained Archimedes's level up to the Renaissance.

> *A wealthy (15th-century) German merchant, seeking to provide his son with a good business education, consulted a learned man as to which European institution offered the best training. "If you only want him to be able to cope with addition and subtraction," the expert replied, "then any French or German university will do. But if you are intent on your son going on to multiplication and division—assuming that he has sufficient gifts—then you will have to send him to Italy.*[1]

By the 19th century, Archimedes had finally been outstripped both as a theorist and as an (applied) numerical analyst, see [7].

> *In 1831, Fourier's posthumous work on equations showed 33 figures of solution, got with enormous labour. Thinking this a good opportunity to illustrate the superiority of the method of W. G. Horner, not yet known in France, and not much known in England, I proposed to one of my classes, in 1841, to beat Fourier on this point, as a Christmas exercise. I received several answers, agreeing with each other, to 50 places of decimals. In 1848, I repeated the proposal, requesting that 50 places might be exceeded: I obtained answers of 75, 65, 63, 58, 57, and 52 places.* (Augustus De Morgan[2])

De Morgan seems to have been one of the first to mistrust William Shanks's epic computations of Pi—to 527, 607, and 727 places [2, pp. 147–161], noting there were too few sevens. But the error was only confirmed three quarters of a century later in 1944 by Ferguson with the help of a calculator in the last pre-computer calculations of π—though until around 1950 a "computer" was still a person and ENIAC was an "Electronic Numerical Integrator and Calculator" [2, pp. 277–281] on which Metropolis and Reitwiesner computed Pi to 2037 places in 1948 and confirmed that there were the expected number of sevens.

Reitwiesner, then working at the Ballistics Research Laboratory, Aberdeen Proving Ground in Maryland, starts his article [2, pp. 277–281] with

> *Early in June, 1949, Professor* JOHN VON NEUMANN *expressed an interest in the possibility that the ENIAC might sometime be employed to determine the value of π and e to many decimal places with a view toward obtaining a statistical measure of the randomness of distribution of the digits.*

The paper notes that e appears to be *too* random—this is now proven—and ends by respecting an oft-neglected "best-practice":

> *Values of the auxiliary numbers* arccot 5 *and* arccot 239 *to 2035D . . . have been deposited in the library of Brown University and the UMT file of MTAC.*

The 20th century's "Top Ten"

The digital computer, of course, greatly stimulated both the appreciation of and the need for algorithms and for algorithmic analysis. At the beginning of this century, Sullivan and Dongarra could write, "Great algorithms are the poetry of computation," when they compiled a list of the 10 algorithms having "the greatest influence on the development and practice of science and engineering in the 20th century".[3] Chronologically ordered, they are:

- #1. 1946: **The Metropolis Algorithm for Monte Carlo**. Through the use of random processes, this algorithm offers an efficient way to stumble toward answers to problems that are too complicated to solve exactly.
- #2. 1947: **Simplex Method for Linear Programming.** An elegant solution to a common problem in planning and decision making.
- #3. 1950: **Krylov Subspace Iteration Method.** A technique for rapidly solving the linear equations that abound in scientific computation.
- #4. 1951: **The Decompositional Approach to Matrix Computations.** A suite of techniques for numerical linear algebra.
- #5. 1957: **The Fortran Optimizing Compiler.** Turns high-level code into efficient computer-readable code.
- #6. 1959: **QR Algorithm for Computing Eigenvalues.** Another crucial matrix operation made swift and practical.

[1]From page 577 of *The Universal History of Numbers: From Prehistory to the Invention of the Computer*, translated from French, John Wiley, 2000.
[2]Quoted by Adrian Rice in "What Makes a Great Mathematics Teacher?" on page 542 of The *American Mathematical Monthly*, June–July 1999.
[3]From "Random Samples," *Science* page 799, February 4, 2000. The full article appeared in the January/February 2000 issue of *Computing in Science & Engineering*.

#7. 1962: **Quicksort Algorithms for Sorting.** For the efficient handling of large databases.

#8. 1965: **Fast Fourier Transform.** Perhaps the most ubiquitous algorithm in use today, it breaks down waveforms (like sound) into periodic components.

#9. 1977: **Integer Relation Detection.** A fast method for spotting simple equations satisfied by collections of seemingly unrelated numbers.

#10. 1987: **Fast Multipole Method.** A breakthrough in dealing with the complexity of n-body calculations, applied in problems ranging from celestial mechanics to protein folding.

I observe that eight of these ten winners appeared in the first two decades of serious computing, and that Newton's method was apparently ruled ineligible for consideration.[4] Most of the ten are multiply embedded in every major mathematical computing package.

Just as layers of software, hardware, and middleware have stabilized, so have their roles in scientific, and especially mathematical, computing. When I first taught the simplex method thirty years ago, the texts concentrated on "Y2K"-like tricks for limiting storage demands. Now serious users and researchers will often happily run large-scale problems in MATLAB and other broad-spectrum packages, or rely on *NAG* library routines embedded in *Maple*.

While such out-sourcing or commoditization of scientific computation and numerical analysis is not without its drawbacks, I think the analogy with automobile driving in 1905 and 2005 is apt. We are now in possession of mature—not to be confused with "error-free"—technologies. We can be fairly comfortable that *Mathematica* is sensibly handling round-off or cancelation error, using reasonable termination criteria and the like. Below the hood, *Maple* is optimizing polynomial computations using tools like Horner's rule, running multiple algorithms when there is no clear best choice, and switching to reduced complexity (Karatsuba or FFT-based) multiplication when accuracy so demands. Wouldn't it be nice, though, if all vendors allowed as much peering under the bonnet as *Maple* does!

Example 2. The number of *additive partitions* of n, $p(n)$, is generated by

$$(1.2) \qquad P(q) = 1 + \sum_{n\geq 1} p(n)q^n = \prod_{n\geq 1} (1 - q^n)^{-1}.$$

Thus $p(5) = 7$, because

$$5 = 4 + 1 = 3 + 2 = 3 + 1 + 1 = 2 + 2 + 1$$
$$= 2 + 1 + 1 + 1 = 1 + 1 + 1 + 1 + 1,$$

as we ignore "0" and permutations. Additive partitions are less tractable than multiplicative ones, for there is no analogue of unique prime factorization nor the corresponding structure. Partitions provide a wonderful example of why Keith Devlin calls mathematics "the science of patterns."

Formula (1.2) is easily seen by expanding $(1 - q^n)^{-1}$ and comparing coefficients. A modern computational temperament leads to

Question: How hard is $p(n)$ to compute—in 1900 (for MacMahon the "father of combinatorial analysis") or in 2000 (for *Maple* or *Mathematica*)?

Answer: The computation of $p(200) = 3972999029388$ took MacMahon months and intelligence. Now, however, we can use the most naïve approach: Computing 200 terms of the series for the inverse product in (1.2) instantly produces the result, using either *Mathematica* or *Maple*. Obtaining the result $p(500) = 2300165032574323995027$ is not much more difficult, using the *Maple* code

```
N : =500; coeff(series(1/product
            (1-q^n,n=1..N+1),q,N+1),q,N);
```

Euler's Pentagonal number theorem

Fifteen years ago computing $P(q)$ in *Maple*, was very slow, while taking the series for the reciprocal $Q(q) = \Pi_{n\geq 1}(1 - q^n)$ was quite manageable! Why? Clearly the series for Q must have special properties. Indeed it is *lacunary*:

$$Q(q) = 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + q^{22} + q^{26} - q^{35} - q^{40} + q^{51} + q^{57} - q^{70} - q^{77} + q^{92} + O(q^{100}). \quad (1.3)$$

This lacunarity is now recognized automatically by *Maple*, so the platform works much better, but we are much less likely to discover Euler's gem:

$$\prod_{n=1}^{\infty} (1 - q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)/2}.$$

If we do not immediately recognize these *pentagonal numbers*, then Sloane's online *Encyclopedia of Integer Sequences*[5] immediately comes to the rescue, with abundant references to boot.

This sort of mathematical computation is still in its reasonably early days, but the impact is palpable—and no more so than in the contest and book under review.

About the Contest

For a generation Nick Trefethen has been at the vanguard of developments in scientific computation, both through his own research, on topics such as pseudo-spectra, and through much thoughtful and vigorous activity in the community. In a 1992 essay "The Definition of Numerical Analysis"[6] Trefethen engagingly demolishes the conventional definition of Numerical Analysis as "the science of rounding errors." He explores how this hyperbolic view emerged, and finishes by writing,

> *I believe that the existence of finite algorithms for certain problems, together with other historical forces, has*

[4]It would be interesting to construct a list of the ten most influential earlier algorithms.

[5]A fine model for of 21st-century databases, it is available at www.research.att.com/~njas/sequences

[6]*SIAM News*, November 1992.

distracted us for decades from a balanced view of numerical analysis. Rounding errors and instability are important, and numerical analysts will always be the experts in these subjects and at pains to ensure that the unwary are not tripped up by them. But our central mission is to compute quantities that are typically uncomputable, from an analytical point of view, and to do it with lightning speed. For guidance to the future we should study not Gaussian elimination and its beguiling stability properties, but the diabolically fast conjugate gradient iteration, or Greengard and Rokhlin's O(N) multipole algorithm for particle simulations, or the exponential convergence of spectral methods for solving certain PDEs, or the convergence in O(N) iterations achieved by multigrid methods for many kinds of problems, or even Borwein and Borwein's[7] *magical AGM iteration for determining 1,000,000 digits of π in the blink of an eye. That is the* heart *of numerical analysis.*

In the January 2002 issue of *SIAM News*, Nick Trefethen, by then of Oxford University, presented ten diverse problems used in teaching *modern* graduate numerical analysis students at Oxford University, the answer to each being a certain real number. Readers were challenged to compute ten digits of each answer, with a $100 prize to be awarded to the best entrant. Trefethen wrote, "If anyone gets 50 digits in total, I will be impressed."

And he was. A total of 94 teams, representing 25 different nations, submitted results. Twenty of these teams received a full 100 points (10 correct digits for each problem). They included the late John Boersma, working with Fred Simons and others; Gaston Gonnet (a Maple founder) and Robert Israel; a team containing Carl Devore; and the authors of the book under review variously working alone and with others. These results were much better than expected, but an originally anonymous donor, William J. Browning, provided funds for a $100 award to each of the twenty perfect teams. The present author, David Bailey,[8] and Greg Fee entered, but failed to qualify for an award.[9]

The ten challenge problems

The purpose of computing is insight, not numbers. (Richard Hamming[10])

The ten problems are:

#1. What is $\lim_{\epsilon \to 0} \int_\epsilon^1 x^{-1} \cos(x^{-1} \log x)dx$?

#2. A photon moving at speed 1 in the x-y plane starts at $t = 0$ at $(x,y) = (1/2, 1/10)$ heading due east. Around every integer lattice point (i, j) in the plane, a circular mirror of radius 1/3 has been erected. How far from the origin is the photon at $t = 10$?

#3. The infinite matrix A with entries $a_{11} = 1$, $a_{12} = 1/2$, $a_{21} = 1/3$, $a_{13} = 1/4$, $a_{22} = 1/5$, $a_{31} = 1/6$, etc., is a bounded operator on ℓ^2. What is $\|A\|$?

#4. What is the global minimum of the function $\exp(\sin(50x)) + \sin(60e^y) + \sin(70 \sin x) + \sin(\sin(80y)) - \sin(10(x + y)) + (x^2 + y^2)/4$?

#5. Let $f(z) = 1/\Gamma(z)$, where $\Gamma(z)$ is the gamma function, and let $p(z)$ be the cubic polynomial that best approximates $f(z)$ on the unit disk in the supremum norm $\| \cdot \|_\infty$. What is $\|f - p\|_\infty$?

#6. A flea starts at (0,0) on the infinite 2-D integer lattice and executes a biased random walk: At each step it hops north or south with probability 1/4, east with probability $1/4 + \epsilon$, and west with probability $1/4 - \epsilon$. The probability that the flea returns to (0,0) sometime during its wanderings is 1/2. What is ϵ?

#7. Let A be the 20000×20000 matrix whose entries are zero everywhere except for the primes $2, 3, 5, 7, \cdots, 224737$ along the main diagonal and the number 1 in all the positions a_{ij} with $|i - j| = 1, 2, 4, 8, \cdots, 16384$. What is the (1,1) entry of A^{-1}?

#8. A square plate $[-1,1] \times [-1,1]$ is at temperature $u = 0$. At time $t = 0$ the temperature is increased to $u = 5$ along one of the four sides while being held at $u = 0$ along the other three sides, and heat then flows into the plate according to $u_t = \Delta u$. When does the temperature reach $u = 1$ at the center of the plate?

#9. The integral $I(\alpha) = \int_0^2 [2 + \sin(10\alpha)]x^\alpha \sin(\alpha/(2 - x)) dx$ depends on the parameter α. What is the value $\alpha \in [0,5]$ at which $I(\alpha)$ achieves its maximum?

#10. A particle at the center of a 10×1 rectangle undergoes Brownian motion (i.e., 2-D random walk with infinitesimal step lengths) till it hits the boundary. What is the probability that it hits at one of the ends rather than at one of the sides?

Answers correct to 40 digits to the problems are available at http://web.comlab.ox.ac.uk/oucl/work/nick.trefethen/hundred.html

Quite full details on the contest and the now substantial related literature are beautifully recorded on Bornemann's Web site

http://www-m8.ma.tum.de/m3/bornemann/challengebook/

which accompanies *The SIAM 100-digit Challenge: A Study In High-accuracy Numerical Computing*, which, for brevity, I shall call *The Challenge*.

About the Book and Its Authors

Success in solving these problems requires a broad knowledge of mathematics and numerical analysis, together with

[7]As in many cases, this eponym is inaccurate, if flattering: it really should be Gauss-Brent-Salamin.

[8]Bailey wrote the introduction to the book under review.

[9]We took Nick at his word and turned in 85 digits! We thought that would be a good enough entry and returned to other activities.

[10]In *Numerical Methods for Scientists and Engineers*, 1962.

significant computational effort, to obtain solutions and ensure correctness of the results. The strengths and limitations of *Maple*, *Mathematica*, MATLAB (The *3Ms*), and other software tools such as PARI or GAP, are strikingly revealed in these ventures. Almost all of the solvers relied in large part on one or more of these three packages, and while most solvers attempted to confirm their results, there was no explicit requirement for proofs to be provided. In December 2002, Keller wrote:

> *To the Editor:*
> *Recently, SIAM News published an interesting article by Nick Trefethen (July/August 2002, page 1) presenting the answers to a set of problems he had proposed previously (January/February 2002, page 1). The answers were computed digits, and the clever methods of computation were described.*
> *I found it surprising that no proof of the correctness of the answers was given. Omitting such proofs is the accepted procedure in scientific computing. However, in a contest for calculating precise digits, one might have hoped for more.*
> *Joseph B. Keller, Stanford University*

In my view Keller's request for proofs as opposed to compelling evidence of correctness is, in this context, somewhat unreasonable, and even in the long term counterproductive [3, 4]. Nonetheless, the authors of *The Challenge* have made a complete and cogent response to Keller and much much more. The interest generated by the contest has with merit extended to *The Challenge*, which has already received reviews in places such as *Science*, where mathematics is not often seen.

Different readers, depending on temperament, tools, and training, will find the same problem more or less interesting and more or less challenging. The book is arranged so the ten problems can be read independently. In all cases multiple solution techniques are given; background, mathematics, implementation details—variously in each of the *3Ms* or otherwise—and extensions are discussed, all in a highly readable and engaging way.

Each problem has its own chapter with its own lead author. The four authors, Folkmar Bornemann, Dirk Laurie, Stan Wagon, and Jörg Waldvogel, come from four countries on three continents and did not know each other as they worked on the book, though Dirk did visit Jörge and Stan visited Folkmar as they were finishing their manuscript. This illustrates the growing power of the collaboration, networking, and the grid—both human and computational.

Some high spots

As we saw, Joseph Keller raised the question of proof. On careful reading of the book, one may discover proofs of correctness for all problems except for #1, #3, and #5. For problem #5, one difficulty is to develop a robust interval implementation for both complex number computation and, more importantly, for the *Gamma function*. While error bounds for #1 may be out of reach, an analytic solution to #3 seems to this reviewer tantalizingly close.

The authors ultimately provided 10,000-digit solutions to nine of the problems. They say that this improved their knowledge on several fronts as well as being "cool." When using Integer Relation Methods, ultrahigh precision computations are often needed [3]. One (and only one) problem remains totally intractable[11]—at press time, getting more than 300 digits for #3 was impossible.

Some surprises

According to the authors,[12] they were surprised by the following, listed by problem:

#1. The best algorithm for 10,000 digits was the trusty *trapezoidal rule*—a not uncommon personal experience of mine.

#2. Using *interval arithmetic* with starting intervals of size smaller than 10^{-5000}, one can still find the position of the particle at time 2000 (not just time ten), which makes a fine exercise for very high-precision interval computation.

#4. Interval analysis algorithms can handle similar problems in higher dimensions. As a foretaste of future graphic tools, one can solve this problem using current *adaptive 3-D plotting* routines which can catch all the bumps. As an optimizer by background, this was the first problem my group solved using a damped Newton method.

#5. While almost all canned optimization algorithms failed, *differential evolution*, a relatively new type of evolutionary algorithm, worked quite well.

#6. This problem has an almost-closed form in terms of elliptic integrals and leads to a study of random walks on hypercubic lattices, and Watson integrals [3, 4, 5].

#9. The maximum parameter is expressible in terms of a *MeijerG function*. While this was not common knowledge among the contestants, *Mathematica* and *Maple* both will figure this out. This is another measure of the changing environment. It is usually a good idea—and not at all immoral—to data-mine[13] and find out what your favourite one of the *3Ms* knows about your current object of interest. For example, Maple tells one that:

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[11]If only by the authors' new gold standard of 10,000 digits.

[12]Stan Wagon, private communication.

[13]By its own count, Wal-Mart has 460 terabytes of data stored on Teradata mainframes, made by NCR, at its Bentonville headquarters. To put that in perspective, the Internet has less than half as much data . . . ," Constance Hays, "What Wal-Mart Knows About Customers' Habits," *New York Times*, Nov. 14, 2004. Mathematicians also need databases.

```
    The Meijer G function is defined by the inverse
                        Laplace transform
    MeijerG([as,bs],[cs,ds],z)
                  /
                  |
          1       |  GAMMA(1-as+y) GAMMA(cs-y)   y
    = --------    |  ---------------------------  z  dy
      2 Pi I      0  GAMMA(bs-y) GAMMA(1-ds+y)
                  |
                  |
                 /
                  L
  where
    as = [a1, · · · ,am],  GAMMA(1-as+y) = GAMMA(1-a1+y) · · · GAMMA(1-am+y)
    bs = [b1, · · · ,bn],  GAMMA(bs-y) = GAMMA(b1-y)     · · · GAMMA(bn-y)
    cs = [c1, · · · ,cp],  GAMMA(cs-y) = GAMMA(c1-y)     · · · GAMMA(cp-y)
    ds = [d1, · · · dq],   GAMMA(1-ds+y) = GAMMA(1-d1+y) · · · GAMMA(1-dq+y)
```

Another excellent example of how packages are changing mathematics is the *Lambert W function* [4], whose properties and development are very nicely described in a recent article by Brian Hayes [8], *Why W?*

Two big surprises

I finish this section by discussing in more detail the two problems whose resolution most surprised the authors.

The essay on Problem #7, whose principal author was Bornemann, is titled: "Too Large to be Easy, Too Small to Be Hard." Not so long ago a 20,000 × 20,000 matrix was large enough to be hard. Using both *congruential* and *p-adic* methods, Dumas, Turner, and Wan obtained a fully *symbolic* answer, a rational with a 97,000-digit numerator and like denominator. Wan has reduced the time to obtain this to about 15 minutes on one machine, from using many days on many machines. While *p-adic* analysis is susceptible to parallelism, it is less easily attacked than are congruential methods; the need for better parallel algorithms lurks below the surface of much modern computational mathematics.

The surprise here, though, is not that the solution is rational, but that it can be explicitly constructed. The chapter, like the others, offers an interesting menu of numeric and exact solution strategies. Of course, in any numeric approach *ill-conditioning* rears its ugly head, while *sparsity* and other core topics come into play.

My personal favourite, for reasons that may be apparent, is:

Problem #10: "Hitting the Ends." Bornemann starts the chapter by exploring *Monte-Carlo methods*, which are shown to be impracticable. He then reformulates the problem *deterministically* as the value at the center of a 10 × 1 rectangle of an appropriate harmonic measure of the ends, arising from a 5-point discretization of Laplace's equation with Dirichlet boundary conditions. This is then solved by a well-chosen *sparse Cholesky* solver. At this point a reliable numerical value of $3.837587979 \cdot 10^{-7}$ is obtained. And the posed problem is solved numerically to the requisite 10 places.

But this is only the warm-up. We proceed to develop two analytic solutions, the first using *separation of variables* on the underlying PDE on a general $2a \times 2b$ rectangle. We learn that

$$(3.4)\qquad p(a,b) = \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\operatorname{sech}\left(\frac{\pi(2n+1)}{2}\rho\right)$$

where $\rho := a/b$. A second method using *conformal mappings* yields

$$(3.5)\qquad \operatorname{arccot}\rho = p(a,b)\,\frac{\pi}{2} + \arg K(e^{ip(a,b)\pi}),$$

where K is the *complete elliptic integral* of the first kind. It will not be apparent to a reader unfamiliar with inversion of elliptic integrals that (3.4) and (3.5) encode the same solution; but they must, as the solution is unique in (0,1); each can now be used to solve for $p = 10$ to arbitrary precision.

Bornemann finally shows that, for far from simple reasons, the answer is

$$(3.6)\qquad p = \frac{2}{\pi}\arcsin(k_{100}),$$

where

$$k_{100} := \left((3-2\sqrt{2})\,(2+\sqrt{5})\,(-3+\sqrt{10})\,(-\sqrt{2}+\sqrt[4]{5})^2\right)^2$$

a simple composition of one arcsin and a few square roots. No one anticipated a closed form like this.

Let me show how to finish up. An apt equation is [5, (3.2.29)] showing that

$$(3.7)\qquad \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\operatorname{sech}\left(\frac{\pi(2n+1)}{2}\rho\right) = \frac{1}{2}\arcsin k,$$

exactly when $k = k_{\rho^2}$ is parametrized by *theta functions* in terms of the so-called *nome*, $q = \exp(-\pi\rho)$, as Jacobi discovered. We have

$$(3.8)\qquad k_{\rho^2} = \frac{\theta_2^2(q)}{\theta_3^2(q)} = \frac{\sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}}{\sum_{n=-\infty}^{\infty} q^{n^2}}$$

Comparing (3.7) and (3.4), we see that the solution is

$k_{100} = 6.02806910155971082882540712292\ldots \cdot 10^{-7}$,