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What is $\lim_{\epsilon \to 0} \int_{\epsilon}^{1} x^{-1} \cos(x^{-1} \log x)\, dx$?

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A photon moving at speed 1 in the x–y plane starts at time $t = 0$ at $(x, y) = (1/2, 1/10)$ heading due east. Around every integer lattice point (i, j) in the plane, a circular mirror of radius $1/3$ has been erected. How far from $(0, 0)$ is the photon at $t = 10$?

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The infinite matrix A with entries $a_{11} = 1, a_{12} = 1/2, a_{21} = 1/3, a_{13} = 1/4, a_{22} = 1/5, a_{31} = 1/6$, and so on, is a bounded operator on ℓ^2. What is $\|A\|$?

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What is the global minimum of the function

$$e^{\sin(50x)} + \sin(60e^y) + \sin(70 \sin x) + \sin(\sin(80y)) - \sin(10(x + y)) + (x^2 + y^2)/4?$$

5 A Complex Optimization **101**

Let $f(z) = 1/\Gamma(z)$, where $\Gamma(z)$ is the gamma function, and let $p(z)$ be the cubic polynomial that best approximates $f(z)$ on the unit disk in the supremum norm $\| \cdot \|_\infty$. What is $\|f - p\|_\infty$?

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A flea starts at $(0, 0)$ on the infinite two-dimensional integer lattice and executes a biased random walk: At each step it hops north or south with probability $1/4$, east with probability $1/4+\epsilon$, and west with probability $1/4-\epsilon$. The probability that the flea returns to $(0, 0)$ sometime during its wanderings is $1/2$. What is ϵ?

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Let A be the $20{,}000 \times 20{,}000$ matrix whose entries are zero everywhere except for the primes $2, 3, 5, 7, \ldots, 224737$ along the main diagonal and the number 1 in all the positions a_{ij} with $|i - j| = 1, 2, 4, 8, \ldots, 16384$. What is the $(1,1)$ entry of A^{-1}?

8 In the Moment of Heat **169**

A square plate $[-1, 1] \times [-1, 1]$ is at temperature $u = 0$. At time $t = 0$ the temperature is increased to $u = 5$ along one of the four sides while being held at $u = 0$ along the other three sides, and heat then flows into the plate according to $u_t = \Delta u$. When does the temperature reach $u = 1$ at the center of the plate?

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The integral $I(\alpha) = \int_0^2 \big(2 + \sin(10\alpha)\big)x^\alpha \sin\big(\alpha/(2 - x)\big)\, dx$ depends on the parameter α. What is the value $\alpha \in [0, 5]$ at which $I(\alpha)$ achieves its maximum?

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A particle at the center of a 10×1 rectangle undergoes Brownian motion (i.e., two-dimensional random walk with infinitesimal step lengths) until it hits the boundary. What is the probability that it hits at one of the ends rather than at one of the sides?

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