Formula (6.16) Revisited: Obtaining the Generating Function by a Holonomic Differential Equation

(or: no "tricks" anymore...)

Folkmar Bornemann, March 9, 2007

We apply Zeilberger's algorithm to the return probabilities p_k to obtain a holonomic recurrence equation. (Writing $a = p_{EW}$ and $b = p_{NS}$ for short.) This is, once more, what we have done in the Maple session preceeding (6.14).

```
> rec:=sumtools[sumrecursion](binomial(2*k,k)*binomial(k,j)^2*a^(2*
  j)*b^(2*(k-j)),j,p(k));
```

$$rec := 4\,(a-b)^2\,(a+b)^2\,(2\,k-1)\,(2\,k-3)\,\mathrm{p}(k-2) - 2\,(a^2+b^2)\,(2\,k-1)^2\,\mathrm{p}(k-1) + \mathrm{p}(k)\,k^2$$

Now, we depart and transform the holonomic recurrence for p_k into a holonomic differential equation for the generating function $\mathrm{E}(x) = \sum_{k=0}^{\infty} p_k\, x^k$.

```
> dic:=gfun[rectodiffeq]({rec=0,p(0)=1,p(1)=2*(a^2+b^2)},p(k),E(x))
```

```
  : deq:=remove(hastype,dic,`=`); ic:=select(hastype,dic,`=`);
```

$$deq := \{(12\,x\,a^4 - 24\,x\,b^2\,a^2 + 12\,x\,b^4 - 2\,a^2 - 2\,b^2)\,\mathrm{E}(x) + (-96\,x^2\,b^2\,a^2 + 48\,x^2\,a^4 + 48\,x^2\,b^4 - 16\,x\,a^2 - 16\,x\,b^2 + 1)\left(\frac{d}{dx}\mathrm{E}(x)\right) + (16\,x^3\,b^4 - 32\,x^3\,b^2\,a^2 + 16\,x^3\,a^4 - 8\,x^2\,b^2 - 8\,x^2\,a^2 + x)\left(\frac{d^2}{dx^2}\mathrm{E}(x)\right)\}$$

$$ic := \{\mathrm{D}(E)(0) = 2\,a^2 + 2\,b^2, \mathrm{E}(0) = 1\}$$

The differential equation degenerates for $x = 0$, but the initial value $\frac{d}{dx}\mathrm{E}(0) = 2\,(a^2 + b^2)$ is by construction compatible with this degeneracies.

```
> subs(x=0,deq);
```

$$\{(-2\,a^2 - 2\,b^2)\,\mathrm{E}(0) + \mathrm{diff}(\mathrm{E}(0), 0)\}$$

The second order holonomic differential equation is easily solved *numerically* for the desired expected value E of the number of visits by integrating it to $x = 1$. This yields a further numerical method to solve Problem 6 in less than a second (see the accompanying Matlab routine 'problem6_by_diffeq.m'). However, we could courageously try to find the solution to it using *symbolic* methods.

```
> sol:=DEtools[hypergeomsols](deq,E(x));
```

$$sol := \left[\frac{\mathrm{LegendreP}\left(\frac{-1}{2}, \frac{24\,x\,b\,a+4\,x\,a^2+4\,x\,b^2-1}{4\,x\,a^2-8\,x\,b\,a+4\,x\,b^2-1}\right)}{\sqrt{4\,x\,a^2-8\,x\,b\,a+4\,x\,b^2-1}}, \frac{\mathrm{LegendreQ}\left(\frac{-1}{2}, \frac{24\,x\,b\,a+4\,x\,a^2+4\,x\,b^2-1}{4\,x\,a^2-8\,x\,b\,a+4\,x\,b^2-1}\right)}{\sqrt{4\,x\,a^2-8\,x\,b\,a+4\,x\,b^2-1}}\right]$$

Because of the degeneracy, there is a one-dimensional subspace of solutions that have finite values at $x = 0$. Lets check the given basis elements for that property.

```
> eval(subs(x=0,sol[1]));
```

$$-I$$

```
> eval(subs(x=0,sol[2]));
Error, (in LegendreQ) numeric exception: division by zero
```

In half of the runs (Maple behaves non-deterministic here), we are lucky and the first element of the given basis, involving the *LegendreP* function, is finite at $x = 0$. (Otherwise restart and repeat.) We normalize for $\mathrm{E}(0) = 1$.

```
> sol[1]/eval(subs(x=0,sol[1]));
```

$$\frac{\mathrm{LegendreP}\left(\frac{-1}{2}, \frac{24\,x\,b\,a+4\,x\,a^2+4\,x\,b^2-1}{4\,x\,a^2-8\,x\,b\,a+4\,x\,b^2-1}\right) I}{\sqrt{4\,x\,a^2-8\,x\,b\,a+4\,x\,b^2-1}}$$

Cross-check that the differential equation is satisfied.

```
> simplify(subs(E(x)=%,deq));
```

$$\{0\}$$

The expression that we have obtained already qualifies as a "closed" form solution. However, we can even do better by recognizing that $\mathrm{LegendreP}\left(-\frac{1}{2}, z\right)$ is related to the complete elliptic integral of the first kind K (see Abramowitz/Stegun §8.13). The desired expected value E of the number of visits is obtained by evaluating at $x = 1$.

```
> E=simplify(subs(x=1,convert(%%,EllipticK))) assuming
  0<=b,b<=a,a-b<=1/2;
```

$$E = \frac{2\,\mathrm{EllipticK}\left(\frac{4\sqrt{b}\,\sqrt{a}}{\sqrt{1-4\,a^2+8\,b\,a-4\,b^2}}\right)}{\sqrt{1-4\,a^2+8\,b\,a-4\,b^2}\;\pi}$$

```
>
```

This is exactly formula (6.16) in §6.5, that we had previously obtained by a "trick" due to Herb Wilf. Our approach here is probably more systematic.