An Experimental Approach to the Singular Modulus $k_{100}$

Folkmar Bornemann, March 1, 2005

Inspired by Jonathan Borwein’s manuscript Review of the SIAM Hundred Digit Challenge Prepared for the Mathematical Intelligencer.

The minimal polynomial of $\sqrt{k_{100}}$

\[
\begin{align*}
<< \text{NumberTheory}'\text{Recognize}' \\
\text{k100SqrtNumerical} &= \text{N[ModularLambda}[\sqrt{-100}]^{1/4}, 50]; \\
p &= \text{Recognize[k100SqrtNumerical, 12, t]} \\
1 - 1288 t + 20 t^2 - 1288 t^3 - 26 t^4 + 1288 t^5 + 20 t^6 + 1288 t^7 + t^8
\end{align*}
\]

Obtaining a Suitable Field Extension

The prime factors of 100 are 2 and 5. Therefore, a look on many of the known expressions for singular moduli suggests to try splitting the minimal polynomial $p$ over a field extension of $\sqrt{2}, \sqrt{5}$. In this ground field itself, the minimal polynomial splits into quadratic factors only. A little experimentation with additional simply radicals of fourth order reveals that $\sqrt{k_{100}} \in \mathbb{Q}[\sqrt{2}, \sqrt{5}]$, though this cannot, as a real field, be the splitting field of $p$ (there are still some quadratic factors left). However, this way we obtain a rather short, though not really beautiful, radical expression for $\sqrt{k_{100}}$:

\[
\sqrt{k_{100}} = (k100SqrtRadical = \text{Factor}[p, \text{Extension} \to \{\sqrt{2}, 5^{1/4}\}] \{3\} / . t \to 0)
\]

\[
\sqrt{k_{100}} = -161 + 114 \sqrt{2} - 108 5^{1/4} + 76 \sqrt{2} 5^{1/4} - 72 \sqrt{5} - 48 5^{3/4} + 34 \sqrt{2} 5^{3/4} + 51 \sqrt{10}
\]

Beautification

The form of the radical expression for $\sqrt{k_{100}}$ suggests that, if there is a factorization into simpler expressions at all, one should look for one of the form \(\text{well, this might be too much of hindsight for being a viable experimental approach…}\

\[
k100SqrtRadicalFactored = a_1 (a_2 + \sqrt{2}) (a_3 + \sqrt{5}) (a_4 + \sqrt{10}) (a_5 \sqrt{2} + 5^{1/4})^2;
\]

Let us give it a try and calculate the coefficients $a_1, \ldots, a_5$.
\[ k_{100} = \left( k_{100SqrtRadicalFactored} / . \text{First}@ \right) \]
\[ \text{Solve}[# = 0 &@\text{Coefficient}[\text{Expand}[k_{100SqrtRadicalFactored}] - \]
\[ k_{100SqrtRadical} / . \{\sqrt{2} 5^{1/4} \rightarrow u_1, \sqrt{2} 5^{3/4} \rightarrow u_2, \]
\[ \sqrt{2} \rightarrow u_3, \sqrt{5} \rightarrow u_4, 5^{1/4} \rightarrow u_5, 5^{3/4} \rightarrow u_6, \sqrt{10} \rightarrow u_7\}, \]
\[ \text{Table}[u_i, \{i, 7\}], \text{Table}[a_i, \{i, 5\}] \] // Simplify \]

\[ k_{100} = (-3 + 2 \sqrt{2})^2 (\sqrt{2} - 5^{1/4})^4 (2 + \sqrt{5})^2 (-3 + \sqrt{10})^2 \]

Quite a success of the experimental approach.