Problem. Riemann’s prime counting function is defined as
\[ R(x) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \text{li}(x^{1/k}), \]
where \( \mu(k) \) is the Möbius function, which is \((-1)^{\rho}\) when \( k \) is a product of \( \rho \) different primes and zero otherwise, and \( \text{li}(x) = \int_0^x \frac{dt}{\log t} \) is the logarithmic integral, taken as a principal value in Cauchy’s sense. What is the largest positive zero \( x_* \) of \( R \)?

Solution

Expanding the logarithmic integral into the series
\[ \text{li}(x) = \gamma + \log \log x + \sum_{k=1}^{\infty} \frac{\log k}{k} x^{k}, \]
changing the order of summation, and using the well known formulas
\[ \sum_{n=1}^{\infty} \frac{\mu(n) \log n}{n} = -1, \quad \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}, \quad s \geq 1, \]
we obtain, as Gram did in 1884, the alternative expression
\[ \begin{align*}
R(x) &= 1 + \sum_{k=1}^{\infty} \frac{\log k}{k \cdot k! \zeta(k+1)}.
\end{align*} \]

A comparison with the Taylor series of the exponential function shows that \( R(e^\tau) \) is entire as a function of \( \tau \).

Lemma. For \( \tau > 0 \) there holds the expansion\(^1\)
\[ R(e^{-\tau}) = \sum_{\rho \in \sigma(\rho)=0} \frac{\Gamma(1-\rho)}{(\rho-1) \zeta(\rho)} \tau^{\rho-1}, \]
where the sum is considered in order of increasing absolute value of the zeros.

Proof. For \( \tau > 0 \) fixed, we consider the meromorphic function
\[ f_\tau(z) = \frac{\pi}{\sin(\pi z)} \frac{\tau^z}{z \Gamma(z+1) \zeta(z+1)} = -\frac{\Gamma(-z) \tau^z}{z \zeta(z+1)}. \]
The equality follows from the reflection formula of the \( \Gamma \) function, i.e., \( \pi / \sin(\pi z) = \Gamma(z) \Gamma(1-z) \). We observe that \( z f_\tau(z) \) tends to zero uniformly in \( \arg z \) when

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\(^1\)We assume that the zeros of \( \zeta \) are simple (no multiple zeros are yet known). This is, however, not an essential assumption, but simplifies the terms of the infinite series running over all zeros.
\[ |z| \to \infty \] in such a way that the zeros of \( \sin(\pi z) \) and \( \zeta(z + 1) \) are avoided uniformly. Consequently, by Cauchy’s residue theorem, the sum of the residues of \( f_\tau(z) \) at all of its poles is zero. The poles of \( f_\tau \) are all simple and they are located at the poles of \( \Gamma(-z) \), i.e., at \( z = 0, 1, 2, \ldots \), and at the values \( z \) for which \( \zeta(z + 1) = 0 \).

The determination of the residues is straightforward: at \( z = k \) with \( k \in \mathbb{N} \) the residue is calculated from the first expression of \( f_\tau \) as
\[
\frac{(-\tau)^k}{k \cdot k! \zeta(k + 1)},
\]
at \( z = 0 \) it is 1 because of \( \lim_{z \to 0} z \zeta(z + 1) = 1 \). Finally at \( \rho - 1 \) with \( \zeta(\rho) = 0 \) the residue is obtained from the second expression of \( f_\tau \) as
\[
\frac{\Gamma(1 - \rho) \tau^{\rho - 1}}{(\rho - 1) \zeta'(\rho)}.
\]

Hence, the asserted results follows. \( \square \)

**Corollary.** Let \( N_\zeta \) denote the set of non-trivial roots of the Riemann \( \zeta \) function, i.e., the roots that are conjectured to have real part \( 1/2 \). For \( \tau > 0 \) there holds the expansion
\[
R(e^{-\tau}) = -\sum_{k=1}^{\infty} \frac{(2k)!}{(2k + 1) \zeta'(-2k)} \tau^{-2k-1} + \sum_{\rho \in N_\zeta} \frac{\Gamma(1 - \rho) \tau^{\rho - 1}}{(\rho - 1) \zeta'(\rho)},
\]
where the last sum is considered in order of increasing absolute value of the zeros.

**Proof.** The trivial zeros of \( \zeta(z) \) are located at \( \rho = -2, -4, -6, \ldots \), and the terms of the series representing \( R \) in the Lemma evaluate as
\[
\left. \frac{\Gamma(1 - \rho) \tau^{\rho - 1}}{(\rho - 1) \zeta'(\rho)} \right|_{\rho = -2k} = \frac{(2k)! \tau^{-2k-1}}{(2k + 1) \zeta'(-2k)}.
\]
Separating the non-trivial roots from the trivial ones gives the desired result. \( \square \)

Now, the point is that the two series of the right-hand side of the Corollary are rapidly convergent for large values of \( \tau \). Using symmetries we can simplify further:
\[
R(e^{-\tau}) = -\sum_{k=1}^{\infty} \frac{(2k)!}{(2k + 1) \zeta'(-2k)} \tau^{-2k-1} + 2 \sum_{\rho \in N_\zeta: \Re(\rho) > 0} \Re \left( \frac{\Gamma(1 - \rho) \tau^{\rho - 1}}{(\rho - 1) \zeta'(\rho)} \right).
\]
For our calculation we used a table for the first 100 non-trivial zeros of \( \zeta \) to 1000-digits precision that Andrew Odlyzko provides at the URL
\[ \text{http://www.dtc.umn.edu/~odlyzko/zeta_tables/}. \]
It turns out, that just one term of the first sum and one term of the second term is enough for about five digits of \( \tau_* = -\log(x_*) \), i.e., for plotting accuracy in Fig. 1. Two terms of the first, four of the second sum are needed for 10 digits of \( x_* \) using IEEE machine precision, 5 terms of the first and 12 of the second sum do the job for 28 digits of \( x_* \). This last calculation took about 5 seconds on a 2 GHz PC using Mathematica 5.0 and led to the following solution:
\[
x_* \approx 1.8286 43269 75252 26104 09732 527 \times 10^{-14828},
\]
\[
\tau_* = -\log x_* \approx 34142.12818 46064 64992 63004 60679 84.
\]
Jörg Waldvogel has obtained a different looking series for $R$ which, however, turns out to be identical to ours. We will derive his formula from ours by transforming our series termwise using the functional equation for the $\zeta$ function

$$\zeta(1 - z) = \frac{2 \cos(\pi z/2) \Gamma(z)}{(2\pi)^z} \zeta(z).$$

Taking the derivative yields at the argument $z = 2k + 1$, $k \in \mathbb{N}$,

(*) $$\zeta'(-2k) = \pi \frac{(-1)^k (2k)!}{(2\pi)^{2k+1}} \zeta(2k + 1),$$

and at the argument $\rho \in N_\zeta$

(**) $$\zeta'(1 - \rho) = -\frac{2 \cos(\pi \rho/2) \Gamma(\rho)}{(2\pi)^\rho} \zeta'(\rho).$$

With $\tau = 2\pi t$, the first equation (*) allows the transformation

$$\frac{(2k)!}{(2k + 1) \zeta'(-2k)} = \frac{(-1)^{k-1} t^{-2k-1}}{\pi(2k + 1)\zeta(2k + 1)}.$$

Next, we observe that because of the functional equation and $\zeta(1/2) \neq 0$ the non-trivial zeros in $N_\zeta$ occur in pairs $(\rho, \rho^*)$ with $\rho^* = 1 - \rho$. (The Riemann Hypothesis asserts that $\rho^* = \pi$.) Therefore, the second equation (**) allows the transformation

$$\frac{\Gamma(1 - \rho^*) \tau^{\rho^*-1}}{(\rho^* - 1) \zeta'(\rho^*)} = \frac{\Gamma(\rho)}{\rho} \frac{(2\pi)^\rho \tau^{-\rho}}{2 \cos(\pi \rho/2) \Gamma(\rho) \zeta'(\rho)} = \frac{t^{-\rho}}{2 \rho \cos(\pi \rho/2) \zeta'(\rho)}.$$

Summarizing, from the expansion given in the Corollary we obtain Jörg Waldvogel’s expansion

$$R(e^{-2\pi t}) = \frac{1}{\pi} \sum_{k=1}^\infty \frac{(-1)^k t^{-2k-1}}{(2k + 1) \zeta(2k + 1)} + \frac{1}{2} \sum_{\rho \in N_\zeta} \frac{t^{-\rho}}{\rho \cos(\pi \rho/2) \zeta'(\rho)},$$

which is valid for $t > 0$. 

Figure 1. $R(x)$ in the vicinity of $x_*$, double logarithmic scale

Appendix

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which is valid for $t > 0$. 

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