SOLUTION OF A PROBLEM POSED BY JÖRG WALDVOGEL

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Problem. *Riemann's prime counting function is defined as*

$$R(x) = \sum_{k=1}^{\infty} \frac{\mu(k)}{k} \operatorname{li}(x^{1/k}),$$

where $\mu(k)$ is the Möbius function, which is $(-1)^\rho$ when k is a product of ρ different primes and zero otherwise, and $\operatorname{li}(x) = \int_0^x dt/\log t$ is the logarithmic integral, taken as a principal value in Cauchy's sense. What is the largest positive zero x_* of R?

Solution

Expanding the logarithmic integral into the series

$$\operatorname{li}(x) = \gamma + \log\log x + \sum_{k=1}^{\infty} \frac{\log^k x}{k \cdot k!},$$

changing the order of summation, and using the well known formulas

$$\sum_{n=1}^{\infty} \frac{\mu(n)\log n}{n} = -1, \qquad \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}, \quad s \geq 1,$$

we obtain, as Gram did in 1884, the alternative expression

$$R(x) = 1 + \sum_{k=1}^{\infty} \frac{\log^k x}{k \cdot k!\, \zeta(k+1)}.$$

A comparison with the Taylor series of the exponential function shows that $R(e^\tau)$ is *entire* as a function of τ.

Lemma. *For $\tau > 0$ there holds the expansion*[1]

$$\boxed{R(e^{-\tau}) = \sum_{\rho:\zeta(\rho)=0} \frac{\Gamma(1-\rho)}{(\rho-1)\,\zeta'(\rho)} \tau^{\rho-1},}$$

where the sum is considered in order of increasing absolute value of the zeros.

Proof. For $\tau > 0$ fixed, we consider the meromorphic function

$$f_\tau(z) = \frac{\pi}{\sin(\pi z)} \frac{\tau^z}{z\,\Gamma(z+1)\,\zeta(z+1)} = -\frac{\Gamma(-z)\,\tau^z}{z\,\zeta(z+1)}.$$

The equality follows from the reflection formula of the Γ function, i.e., $\pi/\sin(\pi z) = \Gamma(z)\Gamma(1-z)$. We observe that $z f_\tau(z)$ tends to zero uniformely in $\arg z$ when

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[1] We assume that the zeros of ζ are simple (no multiple zeros are yet known). This is, however, not an essential assumption, but simplifies the terms of the infinite series running over all zeros.

$|z| \to \infty$ in such a way that the zeros of $\sin(\pi z)$ and $\zeta(z+1)$ are avoided uniformely. Consequently, by Cauchy's residue theorem, the sum of the residues of $f_\tau(z)$ at all of its poles is zero. The poles of f_τ are all simple and they are located at the poles of $\Gamma(-z)$, i.e., at $z = 0, 1, 2, \ldots$, and at the values z for which $\zeta(z+1) = 0$.

The determination of the residues is straightforward: at $z = k$ with $k \in \mathbb{N}$ the residue is calculated from the first expression of f_τ as

$$\frac{(-\tau)^k}{k \cdot k!\, \zeta(k+1)},$$

at $z = 0$ it is 1 because of $\lim_{z\to 0} z\zeta(z+1) = 1$. Finally at $\rho - 1$ with $\zeta(\rho) = 0$ the residue is obtained from the second expression of f_τ as

$$\frac{\Gamma(1-\rho)\, \tau^{\rho-1}}{(1-\rho)\zeta'(\rho)}.$$

Hence, the asserted results follows. □

Corollary. *Let N_ζ denote the set of non-trivial roots of the Riemann ζ function, i.e., the roots that are conjectured to have real part $1/2$. For $\tau > 0$ there holds the expansion*

$$\boxed{R(e^{-\tau}) = -\sum_{k=1}^{\infty} \frac{(2k)!\, \tau^{-2k-1}}{(2k+1)\, \zeta'(-2k)} + \sum_{\rho \in N_\zeta} \frac{\Gamma(1-\rho)\, \tau^{\rho-1}}{(\rho-1)\, \zeta'(\rho)},}$$

where the last sum is considered in order of increasing absolute value of the zeros.

Proof. The trivial zeros of $\zeta(z)$ are located at $\rho = -2, -4, -6, \ldots$, and the terms of the series representing R in the Lemma evaluate as

$$\left.\frac{\Gamma(1-\rho)\, \tau^{\rho-1}}{(\rho-1)\, \zeta'(\rho)}\right|_{\rho=-2k} = -\frac{(2k)!\, \tau^{-2k-1}}{(2k+1)\, \zeta'(-2k)}.$$

Separating the non-trivial roots from the trivial ones gives the desired result. □

Now, the point is that the two series of the right-hand side of the Corollary are rapidly convergent for large values of τ. Using symmetries we can simplify further:

$$R(e^{-\tau}) = -\sum_{k=1}^{\infty} \frac{(2k)!\, \tau^{-2k-1}}{(2k+1)\, \zeta'(-2k)} + 2 \sum_{\rho \in N_\zeta : \Im\rho > 0} \Re\left(\frac{\Gamma(1-\rho)\, \tau^{\rho-1}}{(\rho-1)\, \zeta'(\rho)}\right).$$

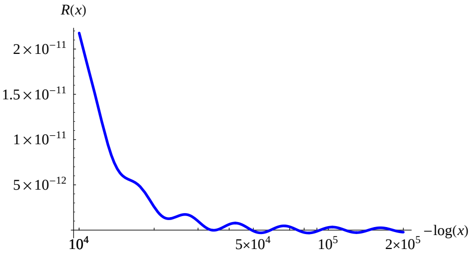
For our calculation we used a table for the first 100 non-trivial zeros of ζ to 1000-digits precision that Andrew Odlyzko provides at the URL

`http://www.dtc.umn.edu/~odlyzko/zeta_tables/` .

It turns out, that just one term of the first sum and one term of the second term is enough for about five digits of $\tau_* = -\log(x_*)$, i.e., for plotting accuracy in Fig. 1. Two terms of the first, four of the second sum are needed for 10 digits of x_* using IEEE machine precision, 5 terms of the first and 12 of the second sum do the job for 28 digits of x_*. This last calculation took about 5 seconds on a 2 GHz PC using *Mathematica 5.0* and led to the following solution:

$$x_* \doteq 1.8286\,43269\,75252\,26104\,09732\,527 \times 10^{-14828},$$

$$\tau_* = -\log x_* \doteq 34142.12818\,46064\,64992\,63004\,60679\,84.$$



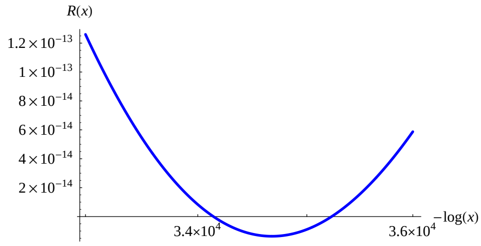


FIGURE 1. $R(x)$ in the vicinity of x_*, double logarithmic scale

APPENDIX

Jörg Waldvogel has obtained a different looking series for R which, however, turns out to be identical to ours. We will derive his formula from ours by transforming our series termwise using the functional equation for the ζ function

$$\zeta(1-z) = \frac{2\cos(\pi z/2)\,\Gamma(z)}{(2\pi)^z}\zeta(z).$$

Taking the derivative yields at the argument $z = 2k+1$, $k \in \mathbb{N}$,

$$(*) \qquad \zeta'(-2k) = \pi \frac{(-1)^k (2k)!}{(2\pi)^{2k+1}}\zeta(2k+1),$$

and at the argument $\rho \in N_\zeta$

$$(**) \qquad \zeta'(1-\rho) = -\frac{2\cos(\pi\rho/2)\Gamma(\rho)}{(2\pi)^\rho}\zeta'(\rho).$$

With $\tau = 2\pi t$, the first equation $(*)$ allows the transformation

$$-\frac{(2k)!\,\tau^{-2k-1}}{(2k+1)\,\zeta'(-2k)} = \frac{(-1)^{k-1}\,t^{-2k-1}}{\pi(2k+1)\zeta(2k+1)}.$$

Next, we observe that because of the functional equation and $\zeta(1/2) \neq 0$ the non-trivial zeros in N_ζ occur in pairs $(\rho, \rho^\star)$ with $\rho^\star = 1-\rho$. (The Riemann Hypothesis asserts that $\rho^\star = \overline{\rho}$.) Therefore, the second equation $(**)$ allows the transformation

$$\frac{\Gamma(1-\rho^\star)}{(\rho^\star-1)}\frac{\tau^{\rho^\star-1}}{\zeta'(\rho^\star)} = \frac{\Gamma(\rho)}{\rho}\frac{(2\pi)^\rho\,\tau^{-\rho}}{2\cos(\pi\rho/2)\,\Gamma(\rho)\,\zeta'(\rho)} = \frac{t^{-\rho}}{2\rho\cos(\pi\rho/2)\,\zeta'(\rho)}.$$

Summarizing, from the expansion given in the Corollary we obtain Jörg Waldvogel's expansion

$$\boxed{R(e^{-2\pi t}) = \frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\,t^{-2k-1}}{(2k+1)\,\zeta(2k+1)} + \frac{1}{2}\sum_{\rho\in N_\zeta}\frac{t^{-\rho}}{\rho\cos(\pi\rho/2)\,\zeta'(\rho)},}$$

which is valid for $t > 0$.

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