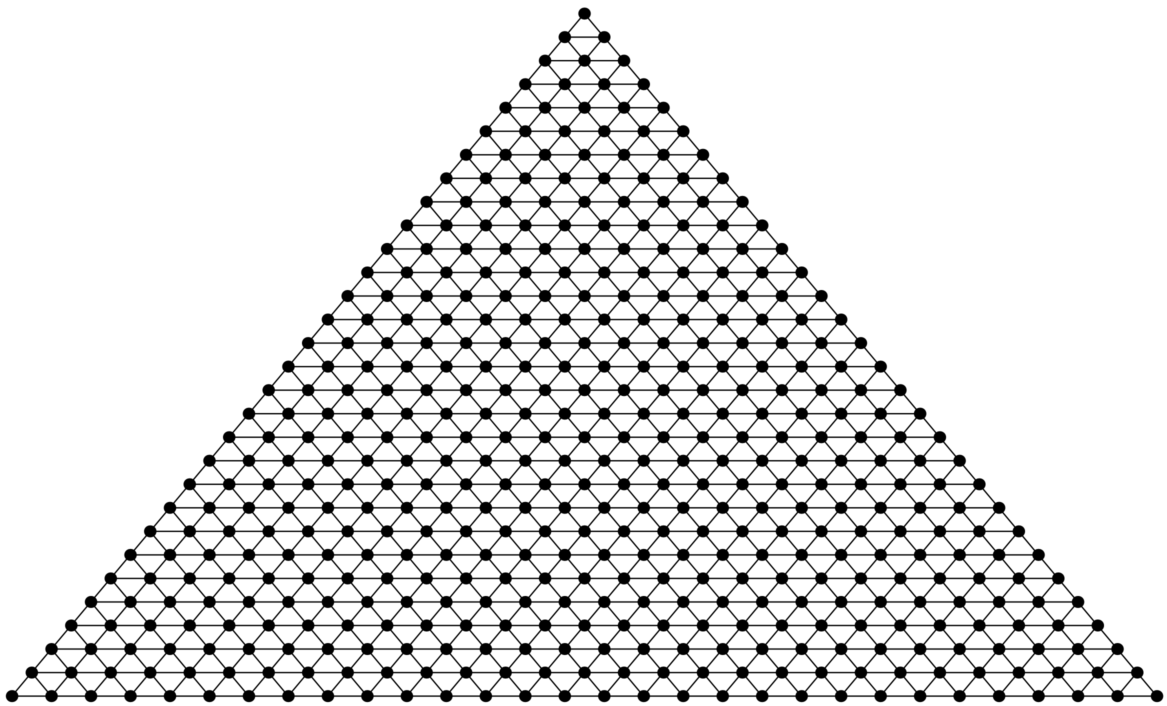
Problem:

A particle starts at the top vertex of the array shown in the figure with 30 points on each side, and then takes 60 random steps. What is the probability that it ends up in the bottom row?



Solution:

There are three kinds of lattice points.

(1) Corner points are connected to only two other points in the lattice, so a particle at one corner will step next to one of the two adjoining points, each with probability 1/2.

(2) Edge points are connected to four other points in the lattice, so a particle at an edge point will step next to one of the four adjoining points, each with probability 1/4.

(3) Interior points are connected to six other points in the lattice, so a particle at an interior point will step next to one of the six adjoining points, each with probability 1/6.

If we needed only a few digits for the solution, then a Monte-Carlo simulation could be used, where random numbers are used to make many random walks. But since the number of grid points and the number of steps are small, we can do better by directly computing an array of probabilities for the position of the particle.

Construct a triangular array of numbers for which the value of each array element is the probability of the particle being in the corresponding location in the lattice after a certain number of steps. Then to find the probability of the particle being in each lattice location after the next random step, split the value at each location into 2, 4, or 6 equal pieces and add them to each of the 2, 4, or 6 neighbor values.

For example, here are the first several steps on a triangular lattice with 5 points on each side. To begin, the point is at the top, so that array element has probability 1 and all the other points are zero.

$$P_0 = \begin{pmatrix} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 0 & & 0 & & 0 & & 0 \end{pmatrix}$$

Here there are 3 corner points, 9 edge points, and 3 interior points. To compute P_1, the probability of the particle being at each lattice location after one random step, first initialize P_1 to zero, then loop over the corner points adding 1/2 the value in P_0 to each of the two neighboring points in P_1. Next for each edge point add 1/4 the value in P_0 to each of its four neighboring points in P_1, and finally for each interior point add 1/6 the value in P_0 to each of its six neighboring points in P_1. This gives

$$P_1 = \begin{pmatrix} & & & & 0 & & & & \\ & & & \frac{1}{2} & & \frac{1}{2} & & & \\ & & 0 & & 0 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 0 & & 0 & & 0 & & 0 \end{pmatrix}$$

After one random step, the particle has 1/2 probability of being in each of the two lattice points in row 2. Now this same process can be iterated to get the array of probabilities for the position of the particle after steps 2, 3, 4, and so on.

$$P_2 = \begin{pmatrix} & & & & \frac{1}{4} & & & & \\ & & & \frac{1}{8} & & \frac{1}{8} & & & \\ & & \frac{1}{8} & & \frac{1}{4} & & \frac{1}{8} & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 0 & & 0 & & 0 & & 0 \end{pmatrix}$$

The 1/4 in the middle of row 3 corresponds to the sum of the probabilities for the two different paths the particle could have taken to get there. The first path is down-left followed by down-right for the two steps, with probability $(1/2)(1/4) = 1/8$. The second path is down-right then down-left, also with probability 1/8.

$$P_3 = \begin{pmatrix} & & & & \frac{1}{16} & & & & \\ & & & \frac{11}{48} & & \frac{11}{48} & & & \\ & & \frac{7}{96} & & \frac{1}{8} & & \frac{7}{96} & & \\ & \frac{1}{32} & & \frac{7}{96} & & \frac{7}{96} & & \frac{1}{32} & \\ 0 & & 0 & & 0 & & 0 & & 0 \end{pmatrix}$$

$$P_4 = \begin{pmatrix} & & & & \frac{11}{96} & & & & \\ & & & \frac{49}{384} & & \frac{49}{384} & & & \\ & & \frac{113}{1152} & & \frac{101}{576} & & \frac{113}{1152} & & \\ & \frac{35}{1152} & & \frac{17}{288} & & \frac{17}{288} & & \frac{35}{1152} & \\ \frac{1}{128} & & \frac{23}{1152} & & \frac{7}{288} & & \frac{23}{1152} & & \frac{1}{128} \end{pmatrix}$$

When this algorithm is used for 60 steps with a P array having 30 rows, there are only 465 array elements to update each step, with one division and no more than six additions per element. This gives the solution to the problem in under 200,000 arithmetic operations, taking well under one second of computer time. All the operations involve nonnegative numbers, so the computation is stable.

Using the built-in 16 significant digit double precision arithmetic gave this result for the sum of the probabilities on the last row after 60 steps (all but the last digit shown are correct).

$9.51234350207431 \times 10^{-6}$

In fact, since the number of operations is fairly small, we can do all the calculations using exact rational arithmetic to get the exact probability that the particle ends up in the last row.

$$\frac{3722200777884626618385530906788866022689096963173522895529}{391302183008102676318141068027642364938466415279908207464546304}$$

$$\approx 9.5123435020743320973531347375383316350767310071737 \times 10^{-6}$$