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Exercises 1, 2 and 3(a) are to be handed in on Thursday, 18.11.2010, before the lecture.

Exercise 1 (Backward Stability) ($*$)
Prove that the multiplication of two real numbers $\left(f : \mathbb{R}^2 \to \mathbb{R},\ (x,y) \mapsto xy\right)$ computed by $\hat{f}(x,y) = fl(x) \odot fl(y)$ is backward stable ($\odot$ denotes the floating point number multiplication).

Exercise 2 (Characteristic Polynomial) ($*$)
The determinant of a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{C}^{2\times 2}$ is given by $det(A) := ad - bc$.
The characteristic polynomial χ_A of A, $\chi_A : \mathbb{C} \to \mathbb{C}$, is defined by

$$\chi_A(z) := det\left(A - zI_2\right) = det\left(\begin{pmatrix} a - z & b \\ c & d - z \end{pmatrix}\right).$$

(Note, that $\chi_A(z)$ truely is a polynomial in z.)
The eigenvalues of A are exactly the roots of χ_A.

(a) Consider the functions $f_j : \mathbb{C}^{2\times 2} \to \mathbb{C}, A \mapsto \lambda_j$ ($\lambda_j = j$th eigenvalue of A).
The relative condition number of f_j for a symmetric matrix with only non-degenerate eigenvalues (i.e. $\lambda_i \neq \lambda_j$ for $i \neq j$) is given by $\kappa^{rel}_{f_j} = |\lambda_{max}|/|\lambda_j|$.
Compute these condition numbers for $A = diag([1 + 10^{-14}, 1])$.

(b) Determine the eigenvalues of the upper matrix A by computing the roots of χ_A (using Matlab). Is this algorithm stable?

Exercise 3 (Variance) ($*$ - please hand in only part (a))
For a vector of real numbers $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ the variance of x is defined by

$$V(x) := \frac{1}{n}\sum_{i=1}^{n}\left(x_i - \frac{1}{n}\sum_{j=1}^{n} x_j\right)^2.$$

(a) Prove the equation

$$V(x) = \frac{1}{n}\left(\left(\sum_{i=1}^{n} x_i^2\right) - \frac{1}{n}\left(\sum_{j=1}^{n} x_j\right)^2\right).$$

(b) Discuss the advantages and disadvantages of the algorithms using the two formulas for the computation of $V(x)$.

(c) Find a vector $x \in \mathbb{R}^n$ for which the first algorithm gives a better result and a vector $y \in \mathbb{R}^n$ for which the second algorithm gives a better result.