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Exercises 1, 2 and 4 are to be handed in on Thursday, 18.11.2010, before the lecture.

Exercise 1 (Matrix Norm) ($*$)

(a) Let $\|.\|$ be a norm on $\mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$. Prove that the following two definitions for the associated matrix norm are equivalent:

$$\|A\| := \max_{\substack{v\in\mathbb{R}^n,\\ v\neq 0}} \frac{\|Av\|}{\|v\|} \quad \textit{and} \quad \|A\| := \max_{\substack{v\in\mathbb{R}^n,\\ \|v\|=1}} \|Av\|.$$

(b) Let $D = diag(d_1, \dots, d_n)$ be a diagonal $n \times n$-matrix and $d := max\{|d_1|, \dots, |d_n|\}$. Show that the matrix norm associated to the norm $\|.\|_2$ on $\mathbb{R}^n$ has the property $\|D\|_2 = d$.

Hint: Recall the definition $\|v\|_2 := \left(\sum_{k=1}^n |v|_k^2\right)^{\frac{1}{2}}$ and prove $\|D\|_2 \geq d_j$ for all j by finding suitable vectors $v \in \mathbb{R}^n$ and $\|D\|_2 \leq d$.

Exercise 2 (Maximum Norm) ($*$)

Consider the maximum norm on $\mathbb{R}^n$ defined by $\|v\|_\infty := \max_{i=1,\dots,n} |v_i|$.

Prove that the associated matrix norm is given by $\|A\| = \max_{i=1,\dots,n} \sum_{j=1}^n |a_{ij}|$

(where $A = (a_{ij})_{\substack{i=1,\dots,n\\ j=1,\dots,n}}$ is an $n \times n$-matrix).

Why is it called the maximum row sum norm?

Hint: Using the first definition from exercise 1 prove that $\|A\| \geq \sum_{j=1}^n |a_{ij}|$ for each i by finding suitable vectors $v \in \mathbb{R}^n$ and $\|A\| \leq \max_{i=1,\dots,n} \sum_{j=1}^n |a_{ij}|$.

Exercise 3 (Wilkinson Polynomial)

Consider the Wilkinson polynomial $w(x) := \prod_{k=1}^{20}(x-j) = a_{20}x^{20} + \dots + a_1 x + a_0$, which obviously has the roots $x_j = j$, $j = 1, \dots, 20$. You can find the (approximative) coefficients $a_{20}, \dots, a_0$ by using the Matlab function $poly(r)$ where $r = [1:20]$ is the vector of the roots. Now perturb the coefficients and plot the new (complex) roots in the complex plane:

(a) do not perturb at all,

(b) multiply a_0 by $(1 + 10^{-5})$,

(c) multiply a_{10} by $(1 + 10^{-5})$,

(d) multiply a_{15} by $(1 + 10^{-5})$.

You can use the Matlab-function $roots(p)$, where p is the vector $[a_{20}, \dots, a_0]$. Please pay attention to the order of the coefficients a_i. Use different colours for the plots of (a),(b),(c),(d).

Exercise 4 (Unitary Matrices) ($*$)

(a) Consider the matrices $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times r}$. Show the equation $(AB)^T = B^T A^T$ (the transpose C^T of a matrix C is defined componentwise by $(C^T)_{ij} := C_{ji}$).

(b) Consider the standard scalar product on $\mathbb{C}^n$ defined by $\langle v, w\rangle := \overline{v}^T w = \sum_{k=1}^n \overline{v}_k w_k$ (for $v, w \in \mathbb{C}^n$) and let $A \in \mathbb{C}^{n\times n}$. Prove the formula $\langle Av, w\rangle = \langle v, A^* w\rangle$, where $A^* := \overline{A}^T$ is the adjoint of A.

Hint: A column vector v can be seen as an $n \times 1$-matrix and its transpose v^T is a row vector and as such just a $1 \times n$-matrix. Use the result from (a).