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Meshfree Methods (MA 5324)
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Vorbemerkung: Alle Antworten sind sehr kurz. Es sind keine längeren Rechnungen notwendig.

Problem 1. Consider two linearly independent polynomials p and q on $\mathbb{R}^2$ and two points $x, y \in \mathbb{R}^2$. Is the matrix

$$P = \begin{bmatrix} p(x) & q(x) \\ p(y) & q(y) \end{bmatrix}$$

necessarily regular? Justify your answer.

Problem 2. Consider $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{R}$. Is $\Phi(x) := \phi(\psi(\|x\|^2))$, $x \in \mathbb{R}^s$, a radial function? Justify your answer.

Problem 3. Show that $\Phi(x) = \varphi(\|x\|)$, $x \in \mathbb{R}^2$, with $\varphi(r) = re^{-r^2}$ is not positive definite.

Problem 4. Consider the hat function

$$\varphi(r) = \begin{cases} r+1, & -1 \le r \le 0, \\ 1-r, & 0 \le r \le 1, \\ 0, & \text{otherwise.} \end{cases}$$

Is this function strictly positive definite on $\mathbb{R}$? What about the radialized version $\Phi(x) = \varphi(\|x\|)$ on $\mathbb{R}^s$? If so, for which s?

Problem 5. Let $\varphi, \psi \in C[0,\infty) \cap C^\infty(0,\infty)$ be completely monotone functions. Using the definition, show that $\varphi + \psi$ is completely monotone, too.

Problem 6. Consider $\varphi : [0,\infty) \to \mathbb{R}$. Suppose that $t \mapsto t\varphi(t)$ is integrable and consider $I\varphi : [0,\infty) \to \mathbb{R}$, $I\varphi(r) = \int_r^\infty t\varphi(t)\, dt$. Show that if φ has compact support then so has $I\varphi$.

Problem 7. Consider the set $X = \{x_1, \ldots, x_6\}$, $x_k = (\cos\frac{2\pi k}{6}, \sin\frac{2\pi k}{6})$, $k = 1, \ldots, 6$ (i.e. the vertices of a hexagon on the unit sphere in $\mathbb{R}^2$). Draw a kd-tree associated to X as well as the induced partition of the square $[-1,1]^2$.

Problem 8. Consider the Sobolev space $W_2^2(\mathbb{R})$ on the real line. Suppose that $\Phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a reproducing kernel for W_2^2. Suppose further that for two functions $f, g \in W_2^2$ we have $(f, \Phi(\cdot, x))_\Phi = (g, \Phi(\cdot, x))_\Phi$ for all $x \in \mathbb{R}$. Show that then $f = g$.

Problem 9. Suppose we use the hat function φ from Problem 4 and the corresponding kernel $\Phi(x,y) = \varphi(|x-y|)$ for interpolation. Let $X = \{0,1\} \subset \mathbb{R}$ and let some function $f : \mathbb{R} \to \mathbb{R}$ be given. What are the corresponding cardinal basis functions? What is the value $I_{f,X}(0.5)$ of the associated interpolant $I_{f,X}$ of f on X?