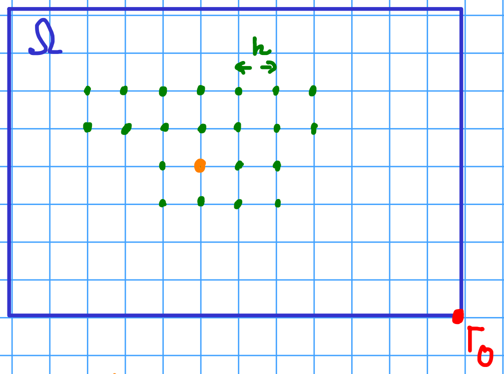
§7 Fast-Marching-Method

(7.1) Discretization of the Eikonal Eq.



$$|\nabla u| = 1/F > 0 \qquad F \text{ velocity-function}$$

$$u|_{\Gamma_0} = 0$$

$\bullet (i,j) \quad i,j \in \mathbb{Z}$

discretization

$$G(u_h) := \max\left(D_x^- u_h, -D_x^+ u_h, 0\right)^2 + \max\left(D_y^- u_h, -D_y^+ u_h, 0\right)^2 - \frac{1}{F^2} \overset{!}{=} 0$$

$u_h : \underbrace{\Omega_h}_{\text{grid points}} \longrightarrow \mathbb{R}$

Written in full:

$$G(u)_{ij} = \max\left(\frac{u_{ij} - u_{i-1,j}}{h}, \frac{u_{ij} - u_{i+1,j}}{h}, 0\right)^2$$

$$+ \max\left(\frac{u_{ij} - u_{i,j-1}}{h}, \frac{u_{ij} - u_{i,j+1}}{h}, 0\right)^2 - \frac{1}{F_{ij}^2} \overset{!}{=} 0$$

nonlinear system of equations of dimension

$$N = \#\Omega_h$$

How to solve?

1st idea: iteration $u_{ij}^0, u_{ij}^1, u_{ij}^2, \dots$

advisible: use structure

Observations:

1) $u_{ij} \uparrow \Longrightarrow G(u)_{ij} \uparrow$ $\quad$ $\uparrow$ means non-decreasing

rest fixed

2) $u_{kl} \uparrow$ $\quad G(u)_{ij} \downarrow$ $\quad$ $\downarrow$ — " — non-increasing

$(k,l) \neq (i,j)$

3) G continuous

4) $G(u)_{ij} \longrightarrow \infty \qquad u_{ij} \longrightarrow \infty$

5) $G(u)_{ij} = -\frac{1}{F_{ij}^2} < 0$ $\quad$ if $\quad u_{ij} = \min(u_{i-1,j}, u_{i+1,j}, u_{i,j-1}, u_{i,j+1})$

(7.2) Nonlinear Gauß-Seidel-Iteration (Rouy-Tourin '92)

A. Start with $u^0 \equiv 0$ $\qquad G(u^0)_{ij} = -\frac{1}{F_{ij}^2} < 0$

B. Solve for all points (i,j) successively

$$G(u)_{ij} = 0 \quad \text{as a function of } u_{ij} \quad \text{with } u_{kl} \quad (k,l) \neq (i,j) \text{ fixed}$$

every substep increases iteration number

$$\begin{pmatrix} u_{11} \\ u_{12} \\ \vdots \end{pmatrix} \longmapsto \begin{pmatrix} G(u)_{11} \\ G(u)_{12} \\ \vdots \end{pmatrix} \overset{!}{=} \begin{pmatrix} 0^{\neq 0} \\ 0 \\ \vdots \end{pmatrix}$$

for every grid point we solve just a scalar equation.

the result of running through all components of u_{ij} gives in the end u^N

C. Desale B.

(7.3) Convergence (as always if there monotonicity)

by induction :

$$\boxed{\begin{array}{ll} 1) & G(u^n)_{ij} \leq 0 \\ 2) & u^n_{ij} \uparrow \quad \text{w.r.t } n \end{array}}$$

proof : $n=0$ ✓

* | by making $G(u)_{ij}=0$ in u_{ij}, G is increasing an therefore u_{ij} must increase
* | by increasing u_{ij} the values of $G(u)_{ke}$ with $(k,l) \neq (i,j)$ is decreasing, so they never get to be > 0.

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from that follows : $u^n \longrightarrow u$ with $G(u)=0$

proof : a) by ① & ② we know that $u^n \longrightarrow u$

why : by ② u^n_{ij} is either $\longrightarrow \infty$ or convergent

then $G(u^n)_{ij} \longrightarrow \infty$ in contradiction to $G(u^n) \leq 0$

b) $G(u^n) \longrightarrow G(u)$ by continuity

$$G(u)_{ij} = \lim_{n\to\infty} G(u^n)_{ij} = 0$$

since in every Gauß-Seidel loop we produce at least once a zero here.

there is subsequence of zeros $\Rightarrow$ $G(u)_{ij}=0$

$$\underbrace{*,*,*,\ldots,*,0,*,*,\ldots,*}_{N}, \underbrace{*,\ldots,*,0,*,\ldots,*}_{N}, \underbrace{*,\ldots,*,0,*,\ldots,*}_{N}, \ldots \longrightarrow G_2(u)_{ij}$$

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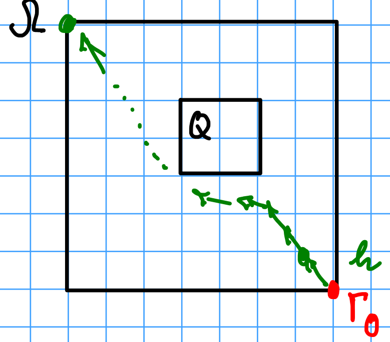
(7.5) Example:



$$F(x) = \begin{cases} 1 & x \notin Q \\ 0.1 & x \in Q \end{cases}$$

$m \times m$ Grid $(N = m^2)$ the # of iterates seem to scale like $c\,m$

why? Information of the propagating front spreads through neighbors only within one outer step (all Gauß-Seidel updates done) the front has only moved by h

but we have to come accross the square: # iterates $\sim \sqrt{2}/h$

observation: Gauß-Seidel is doing much too much work in already converged regions of the solution

(7.4) Solution of $G(u)_{ij} = 0$ in u_{ij} $\longrightarrow$ exercises

(7.6) Discrete causality

imagine you have the solution u of

$$G(u)_{ij} = \max\left(\frac{u_{ij}-u_{i-1,j}}{h}, \frac{u_{ij}-u_{i+1,j}}{h}, 0\right)^2 + \max\left(\frac{u_{ij}-u_{i,j-1}}{h}, \frac{u_{ij}-u_{i,j+1}}{h}, 0\right)^2 - \frac{1}{F_{ij}^2}$$

$$= 0 \quad \forall ij$$

u_{ij} depends only on those neighbouring values, which are $< u_{ij}$. "upwinding"

(corresponds to the picture of solution being the time of first arrival)