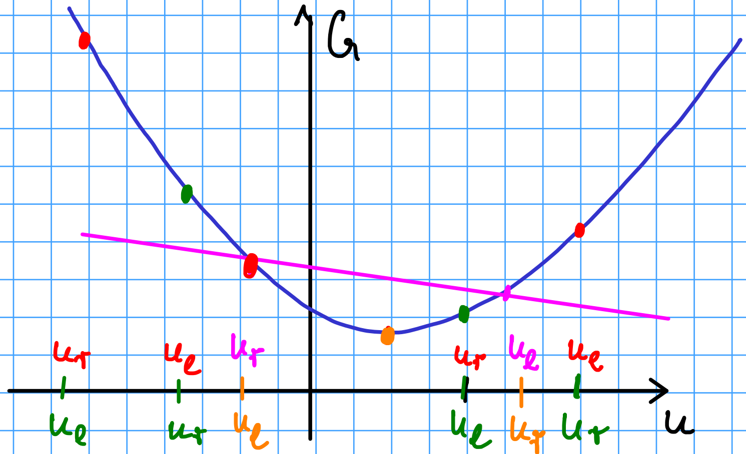
Writing it more compactly:

$$g(u_l, u_r) = G(\omega(0; u_l, u_r)) = \begin{cases} G(u_l) & \text{cases } 1, 4A \\ G(u_r) & \text{—"— } 2, 4B \\ G(u_s) & \text{—"— } 3 \end{cases}$$

$a = G'$



case 1: $a(u_l) > 0 \quad a(u_r) > 0$

$u_l \leq u_r \qquad G(u_l) = \min_{u \in [u_l, u_r]} G(u)$

$u_r \leq u_l \qquad G(u_l) = \max_{u \in [u_l, u_r]} G(u)$

case 2: $a(u_l) < 0 \quad a(u_r) < 0$

$u_l \leq u_r \qquad G(u_r) = \min_{u \in [u_l, u_r]} G(u)$

$u_r \leq u_l \qquad G(u_r) = \max_{u \in [u_l, u_r]} G(u)$

case 3: $a(u_l) < 0 \quad a(u_r) > 0$

$u_l \leq u_r \qquad G(u_s) = \min_{u \in [u_l, u_r]} G(u)$

case 4: $a(u_l) > 0 \quad a(u_r) < 0 \quad \rightarrow u_r \leq u_l$

$$v_{shock} = \frac{G(u_l) - G(u_r)}{u_l - u_r} = \begin{cases} > 0 & A \\ < 0 & B \end{cases}$$

$G(u_r) = \max_{u \in [u_l, u_r]} G(u)$

Same for case A: max

to summarize: Godunov flux can be written in the form (Osher 1982)

$$\boxed{g(u,v) = \operatorname*{ext}_{[u,v]} G = \begin{cases} \min\limits_{[u,v]} G & u \leq v \\ \max\limits_{[u,v]} G & v \leq u \end{cases}}$$

(4.19) Example: Burgers eq.

$$G(u) = \frac{u^2}{2}$$

$u \leq v$
$$g(u,v) = \min_{w \in [u,v]} \frac{w^2}{2} = \frac{1}{2} \max(u,-v,0)^2$$

$u \geq v$
$$g(u,v) = \max_{w \in [u,v]} \frac{w^2}{2} = \frac{1}{2} \max(u,-v,0)^2$$

$$\boxed{g(u,v) = \frac{1}{2} \max(u,-v,0)^2}$$

numerical flux function for Burgers

(4.20) Generalization of Burgers Eq.

$G(u) = \phi(u^2)$, ϕ non-decreasing

Godunov - flux

$$g(u,v) = \underset{w\in[u,v]}{\mathrm{ext}}\, G(w) = \underset{w\in[u,v]}{\mathrm{ext}}\, \phi(w^2) = \phi\left(\underset{w\in[u,v]}{\mathrm{ext}}\, w^2\right)$$

$$= \phi\left(\max(u,-v,0)^2\right)$$

$$\boxed{g(u,v) = \phi(\max(u,-v,0)^2)}$$

$$u_j^{n+1} = u_j^n - \frac{\tau}{h}\left(g(u_j^n, u_{j+1}^n) - g(u_{j-1}^n, u_j^n)\right)$$

(4.21) Remarks on convergence

Godunov method (subject CFL-condition) is monotonic

$$\Updownarrow \quad u_j^n \leq v_j^n \quad (\forall j) \quad \Longrightarrow \quad u_j^{n+1} \leq v_j^{n+1} \quad (\forall j)$$

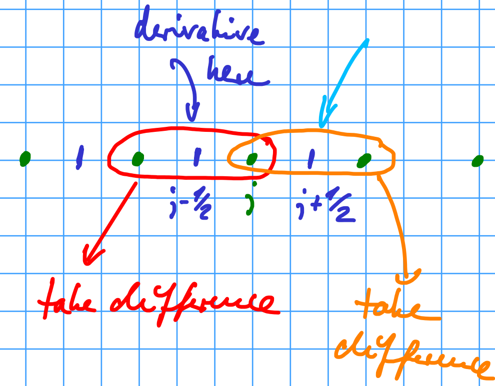
$\Longrightarrow$ (1) for smooth solution we get order of convergence $p=1$

(2) for general solutions ——— " ——— $p = 1/2$

$$\tau/h = \cancel{c}\text{const} \;\&\; \text{CFL condition satisfied}, \qquad \tau \longrightarrow 0$$

$$\|\text{error}\|_{L^1} = \int |\text{pointwise error}|\, dx = O(\tau^p) = \begin{cases} O(\tau) \\ O(\sqrt{\tau}) \end{cases}$$

§5 Discretization of Level-Set & Eikonal Equation



[5.1] What did we learn from conservation laws

$$u_t + H(D_x u) = 0 \qquad x \in \mathbb{R}^1$$

$$u|_{t=0} = u_0$$

expl. Euler in time

$$u_j^{n+1} = u_j^n - \tau H_j^n \qquad H_j^n \approx H(Du(x_j, t_n))$$

proper

recall connection to conservation laws $\quad v = u_x$

$$v_t + H(v)_x = 0$$

$H(v)$ is flux function $\quad \phi(v^2)$

H_j^n should be numerical flux function

$$\boxed{\begin{aligned} H_j^n &= g(v_{j-1/2}^n, v_{j+1/2}^n) \\ &= g(D_x^- u_j^n, D_x^+ u_j^n) \end{aligned}}$$

numerical Hamiltonian

level set eq $\qquad H(p) = F \cdot |p|$

eikonal eq $\qquad H(p) = |p|$

(5.2) Numerical method of Osher–Shu (1992)

$$H(Du) = \phi(|Du|^2)$$

ϕ monotonic (non-decreasing) or anti-monotonic (non-increasing)

using the formula of Osher for flux $\phi(v^2)$ we get

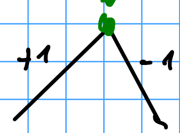
$$\boxed{H_j^n = \phi\left(\max\left(\pm D_x^- u_j^n ,\ \mp D_x^+ u_j^n ,\ 0 \right)^2 \right)}$$

ϕ monotonic

antimonotonic

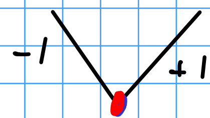
example: eikonal eq $|Du| = 1$

numerical $|Du| = 1$



OK

$|Du| \longrightarrow |\max(D_x^- u_j, -D_x^+ u, 0)|$



NOT OK

numerical $|Du| = 0 \neq 1$

how does this look in multidimensional $x \in \mathbb{R}^d$:

$$u_t + H\left(\frac{\partial}{\partial x_1} u, \dots, \frac{\partial}{\partial x_d} u\right) = 0$$

$$H(p) = \phi(|p|^2) = \phi(p_1^2 + \dots + p_d^2)$$

Osher-Shu 1992

$$\boxed{H_j^n = \phi\left(\sum_{j=1}^{d} \max\left(\pm D_{x_j}^- u, \mp D_{x_j}^+ u, 0\right)^2\right)}$$

ϕ monotonic

antimonotonic

(5.3) Definition : monotonic scheme

$$u^n \leq v^n \implies u^{n+1} \leq v^{n+1}$$

discrete comparison principle

mimicks the property of viscosity solutions $\quad u_0 \leq v_0 \implies u(\cdot,t) \leq v(\cdot,t) \quad \forall t \geq 0$

(5.4) Monotonicity of discrete Hamilton-Jacobi Eq.

Thm :

$$u_j^{n+1} = u_j^n - \tau g\left(D_x^- u_j^n, D_x^+ u_j^n\right) \quad \text{is monotonic}$$

~~if~~

i) $g(w-u, v-w)$ is Lipschitz wrt w with Lipschitz constant L independent of u, v

with respect to

ii) CFL condition : $\boxed{L\,\tau/h \leq 1}$

iii) $g(\uparrow, \downarrow)$ ($\uparrow$ monotonic, $\downarrow$ anti-monotonic)

<u>Proof</u>: to prove : $w - \tau g\left(\frac{w-u}{h}, \frac{v-w}{h}\right)$ is monotonic w.r.t. u, v, w

because of (iii) : $u^{\checkmark}$, $v^{\checkmark}$

let $w \geq \hat{w}$

$$\left(w - \tau g\left(\frac{w-u}{h}, \frac{v-w}{h}\right)\right) - \left(\hat{w} - \tau g\left(\frac{\hat{w}-u}{h}, \frac{v-\hat{w}}{h}\right)\right)$$

$$= w - \hat{w} - \tau\left(g\left(\frac{w-u}{h}, \frac{v-w}{h}\right) - g\left(\frac{\hat{w}-u}{h}, \frac{v-\hat{w}}{h}\right)\right)$$

$$\geq (w-\hat{w}) - \frac{L\tau}{h}\left|\hat{w} - w\right| \qquad \text{by (i)} \qquad w \geq \hat{w}$$

$$= \underbrace{\left(1 - \frac{L\tau}{h}\right)}_{\geq 0 \text{ by CFL-cond.}} \underbrace{(w-\hat{w})}_{\geq 0} \geq 0 \qquad \text{by (ii)} \qquad \boxtimes$$

5.5 Osher-Shu is monotonic for Level-Set-Equation

$$H(Du) = F \cdot |Du|$$

$$g(D_x^- u, D_x^+ u) = \max(F,0) \cdot \max(+D_x^- u, -D_x^+ u, 0) + \min(F,0) \cdot \max(-D_x^- u, +D_x^+ u, 0)$$

Conditions for monotonicity

i) Lipschitz-continuity of $g(w-u, v-w)$ $\quad L \leq \|F\|_\infty$ ✓

ii) CFL condition : $L \tau/h \leq 1$ surely satisfied if $\boxed{\frac{\|F\|_\infty}{h} \tau \leq 1}$ ✓

natural CFL-condition

iii) $g(\uparrow, \downarrow)$ ✓

(5.6) Convergence (Crandall/Lions ~ 1985)

monotonicity $\longrightarrow$ $O(\sqrt{\tau})$ convergence

method of proof : duplication of variables like in the uniqueness proof of viscosity solutions