Analysis III für LB, Übungsblatt 3 – Musterlösung

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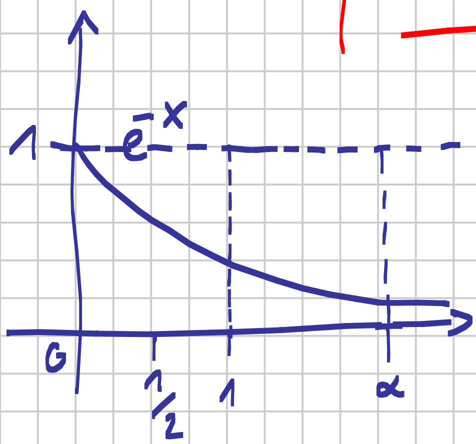
http://www-m3.ma.tum.de

3.1 Zu zeigen ist, dass der Grenzwert

$$\lim_{R\to\infty} \int_0^R e^{-\alpha x^n}\,dx$$

existiert (für alle $n \geq 1$ und $\alpha > 0$).

Es gilt $\boxed{e^{-\alpha x^n} \leq e^{-\alpha x} \quad \text{für} \quad x \in [1,\infty)}$



["denn" für $x = \frac{1}{2}$:

$$e^{-(\frac{1}{2})^n} \geq e^{-\frac{1}{2}},$$

aber für $x \geq 1$:

$$x^n \geq x$$

$$\Rightarrow e^{-x^n} \leq e^{-x}\ ]$$

Damit gilt auch
$$\int_1^R e^{-\alpha x^n}\,dx \leq \int_1^R e^{-\alpha x}\,dx = \left[-\frac{1}{\alpha}e^{-\alpha x}\right]_1^R = \frac{1}{\alpha}e^{-\alpha} - \frac{1}{\alpha}e^{-\alpha R} \leq \frac{1}{\alpha}e^{-\alpha}$$

Damit
$$\lim_{R\to\infty} \int_1^R e^{-\alpha x^n}\,dx \leq \lim_{R\to\infty} \frac{1}{\alpha}e^{-\alpha} = \underline{\underline{\frac{1}{\alpha}e^{-\alpha}}},$$
also

folgt

$$\lim_{R\to\infty} \int_0^R e^{-\alpha x^n}\,dx = \lim_{R\to\infty} \left[\int_0^1 e^{-\alpha x^n}\,dx + \int_1^R e^{-\alpha x^n}\,dx\right]$$

$$= \int_0^1 e^{-\alpha x^n}\,dx + \underbrace{\lim_{R\to\infty} \int_1^R e^{-\alpha x^n}\,dx}_{\leq}$$

$$\leq \int_0^1 1 \, dx + \frac{1}{\alpha} e^{-\alpha} .$$

Falls $f(x) \to 0$ f. $x \to \infty$, existiert dann

$$\int_0^\infty f(x)\,dx \quad \text{immer ?}$$

Nein! Gegenbeispiel: $f(x) = \frac{1}{x}$: $\int_1^R \frac{1}{x}\,dx = \ln R$.

3.2 $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ Setze $s = nt$
$ds = n\,dt$

$$\int_0^\infty e^{-s} \left(\frac{s}{n}\right)^{x-1} \frac{1}{n} ds = \int_0^\infty e^{-s} \left(\frac{1}{n}\right)^{x-1} \cdot \frac{1}{n} s^{x-1} ds$$

$$= n^{-x} \int_0^\infty e^{-s} s^{x-1} ds = n^{-x} \Gamma(x) .$$

3.3

$$\boxed{\begin{gathered} f(a+h) = f(a) + \frac{f'(a)}{1!} h + \frac{f''(a)}{2!} h^2 + \frac{f'''(a)}{3!} h^3 + \dots + R_{n+1}(a+h) \\ R_{n+1}(a+h) = \frac{1}{n!} \int_a^{a+h} (a+h-t)^n f^{(n+1)}(t)\,dt \end{gathered}}$$

Taylorentwicklung für f in 0 ($a=0$, $h=x$, $n=1$):

$$f(x) = f(0) + f'(0)x + \int_0^x (x-t) f''(t)\,dt$$

Integration von f über $[0,a]$:

$$\int_0^a f(x)\,dx = \int_0^a \left[f(0) + f'(0)x + \int_0^x (x-t) f''(t)\,dt \right] dx$$

$$= \underbrace{a f(0) + \frac{a^2}{2} f'(0)}_{= \; I(f)} + \underbrace{\int_0^a \left(\int_0^x (x-t) f''(t)\, dt \right) dx}_{-\,''\,-}$$

d.h. $$\left| \int_0^a f(x)\,dx - I(f) \right| \leq \left| \int_0^a \left(\int_0^x (x-t) f''(t)\,dt \right) dx \right|$$

$$\leq \int_0^a \int_0^x |x-t|\,|f''(t)|\,dt\,dx$$

mit $M = \max_{s\in[0,a]} |f''(s)|$ $$\leq \int_0^a \int_0^x |x-t|\, M\,dt\,dx$$

$$\leq M \int_0^a \left[\int_0^x |x-t|\,dt \right] dx$$

$$= M \int_0^a xt - \frac{1}{2}t^2 \Big|_0^x \, dx$$

$$= M \int_0^a \frac{1}{2} x^2\, dx$$

$$= M \left[\frac{1}{6} x^3 \right]_0^a = \frac{1}{6} a^3 \max_{t\in[0,a]} |f''(t)|$$

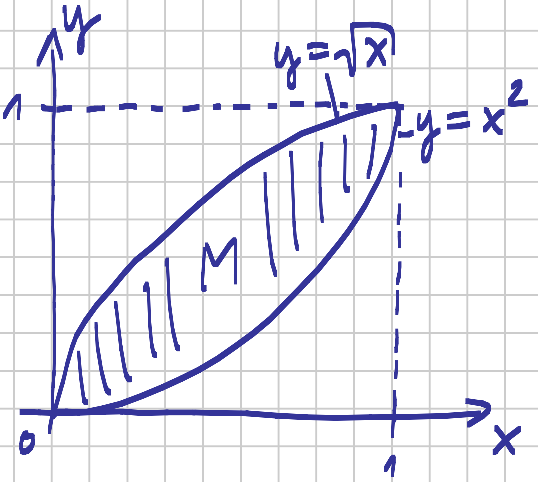
3.4 (i) $f_1(x) =$ Treppenfunktion, z.B. $= \operatorname{sign}(x)$

(ii) f_2 gibt es nicht, f_2 müsste Pol haben in $[-1,1]$, dort wäre sie nicht stetig.

(iii) $f_3(x) = |x|$ stetig, aber nicht diffbar in 0.

(iv) $f_4(x) = x^2 \sin\frac{1}{x}$ ...

3.5



$$\text{Fläche}(M) = \int_0^1 \sqrt{x}\,dx - \int_0^1 x^2\,dx$$

$$= \int_0^1 \sqrt{x} - x^2\,dx = \frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3 \Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \underline{\underline{\frac{1}{3}}}$$