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Research Article

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Singular values and evenness symmetry in random matrix theory

Abstract: Complex Hermitian random matrices with a unitary symmetry can be distinguished by a weight function. When this is even, it is a known result that the distribution of the singular values can be decomposed as the superposition of two independent eigenvalue sequences distributed according to particular matrix ensembles with chiral unitary symmetry. We give decompositions of the distribution of singular values, and the decimation of the singular values – whereby only even, or odd, labels are observed – for real symmetric random matrices with an orthogonal symmetry, and even weight. This requires further specifying the functional form of the weight to one of three types – Gauss, symmetric Jacobi or Cauchy. Inter-relations between gap probabilities with orthogonal and unitary symmetry follow as a corollary. The Gauss case has appeared in a recent work of Bornemann and La Croix. The Cauchy case, when appropriately specialised and upon stereographic projection, gives decompositions for the analogue of the singular values for the circular unitary and circular orthogonal ensembles.

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1 Introduction

The ensembles of real symmetric random matrices $\mathrm{OE}_n(w_1)$ possessing an orthogonal symmetry and of complex Hermitian random matrices $\mathrm{UE}_n(w_2)$ possessing a unitary symmetry are specified by the eigenvalue densities

$$p_\beta(x_1, \dots, x_n) = c_{n,\beta} \prod_{k=1}^{n} w_\beta(x_k) \cdot |\Delta(x_1, \dots, x_n)|^\beta, \quad \beta = 1, 2, \tag{1.1}$$

with some normalization constant $c_{n,\beta}$, each x_k restricted to the interval of support of $w_\beta(x_k)$, and the Vandermonde determinant (note that $\Delta(\xi_1, \dots, \xi_n) \geq 0$ if the arguments are increasingly ordered, $\xi_1 \leq \cdots \leq \xi_n$)

$$\Delta(\xi_1, \dots, \xi_n) = \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \xi_1 & \xi_2 & \cdots & \xi_n \\ \vdots & \vdots & & \vdots \\ \xi_1^{n-1} & \xi_2^{n-1} & \cdots & \xi_n^{n-1} \end{pmatrix} = \prod_{k>j} (\xi_k - \xi_j).$$

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Case	$w_1(x)$	$w_2(x)$	ϖ
Gauss	$e^{-x^2/2}$	e^{-x^2}	∞
Jacobi	$(1-x^2)^a$	$(1-x^2)^{2a+1}$	1
Cauchy	$(1+x^2)^{-(n+a+1)/2}$	$(1+x^2)^{-(n+a)}$	∞

Table 1. Admissible pairs of symmetric weights supported on $(-\varpi, \varpi)$; $a > -1$.

Furthermore, relating to a chiral unitary symmetry, there is the matrix ensemble $\mathrm{chUE}(w_2)$ with *positive* eigenvalues distributed according to the density, see [8, p. 717],

$$p_{\mathrm{ch}}(x_1, \dots, x_n) = c_n^{\mathrm{ch}} \prod_{k=1}^{n} w_2(x_k) \cdot \Delta(x_1^2, \dots, x_n^2)^2. \tag{1.2}$$

As in the theory of orthogonal polynomials, the $w_\beta(x)$ are referred to as weights. In fact, the ensembles are often referred to by the name for the weights used in the theory of orthogonal polynomials. For example, $\mathrm{OE}_N(e^{-x^2/2})$ is referred to as the Gaussian orthogonal ensemble.

In this paper, as a unifying framework for examining eigenvalue properties under evenness symmetry, introduced into random matrix theory in the works [8, 17] and further explored in the Gaussian case in the recent works [3, 7], we study the structure of the singular values of ensembles $\mathrm{OE}_n(w_1)$ with *even* weights w_1 supported on $(-\varpi, \varpi)$ as given in Table 1. The ensemble of singular values will be briefly denoted by $|\mathrm{OE}_n(w_1)|$, in keeping with the relationship between the eigenvalues and singular values – since the ensembles are Hermitian, the singular values are the absolute value of the eigenvalues. Although defined according to the probability density function (1.1), we remark that each ensemble implied by Table 1 can be realised in terms of matrix ensembles defined by a distribution on the elements, see, e.g., [10, Chapters 1–3].

Central to our discussion is the operation of *decimation*, which if applied to $|\mathrm{OE}_n(w_1)|$ results in the two ensembles

$$\text{even}\, |\mathrm{OE}_n(w_1)| \quad \text{and} \quad \text{odd}\, |\mathrm{OE}_n(w_1)|,$$

where we define the even-location decimated ensemble even $|\mathrm{OE}_n(w_1)|$ by taking the second largest, fourth largest etc. singular value, and similarly for odd $|\mathrm{OE}_n(w_1)|$. The results will often depend on the parity μ of the underlying order n and we will, throughout this paper, write

$$n = 2m + \mu, \quad \mu = 0, 1, \quad \hat{m} = m + \mu, \tag{1.3}$$

that is,

$$m = \lfloor n/2 \rfloor, \quad \hat{m} = \lceil n/2 \rceil, \quad \mu = \lceil n/2 \rceil - \lfloor n/2 \rfloor. \tag{1.4}$$

Then, generalizing the corresponding result of Bornemann and La Croix [3, Theorem 1] for Gaussian ensembles, the following structure holds.

Theorem 1.1. *Let w_β, $\beta = 1, 2$, be the weight pairs of the Gauss, symmetric Jacobi or Cauchy case as given in Table 1. Denoting equality of the joint distribution of two ensembles by $\overset{\mathrm{d}}{=}$, there holds*

$$\text{even}\, |\mathrm{OE}_n(w_1)| \overset{\mathrm{d}}{=} \mathrm{chUE}_m(x^{2\mu} w_2), \quad n = 2m + \mu. \tag{1.5}$$

If we recall the superposition representation, see [8, (2.6)],

$$|\mathrm{UE}_n(w_2)| \overset{\mathrm{d}}{=} \mathrm{chUE}_{\hat{m}}(w_2) \cup \mathrm{chUE}_m(x^2 w_2) \tag{1.6}$$

of the singular values of the corresponding unitary ensemble $\mathrm{UE}_n(w_2)$, with both ensembles on the right drawn independently, Theorem 1.1 immediately implies the following remarkable relation between the singular values of $\mathrm{OE}(w_1)$ and $\mathrm{UE}(w_2)$.

Corollary 1.2. *Let w_β, $\beta = 1, 2$, be the weight pairs of the Gauss, symmetric Jacobi or Cauchy case as given in Table 1. Then, with the ensembles on the right drawn independently, there holds*

$$|\mathrm{UE}_n(w_2)| \overset{\mathrm{d}}{=} \text{even}\, |\mathrm{OE}_n(w_1)| \cup \text{even}\, |\mathrm{OE}_{n+1}(w_1)|. \tag{1.7}$$

The superposition (1.7) bears a striking similarity with a corresponding superposition result for the eigenvalue distributions, see [12, pp. 185–186] or [10, Section 6.6], namely,

$$\mathrm{UE}_n(w_2) \overset{\mathrm{d}}{=} \mathrm{even}(\mathrm{OE}_n(w_1) \cup \mathrm{OE}_{n+1}(w_1)).$$

We proceed as follows. First, in Section 2, we give an overview of superposition and decimation results in random matrix theory known from previous studies, so as to properly set the scene for the present study and also as an opportunity to introduce the circular ensembles. In Section 3, a factorized expression for the joint density of the singular values is obtained, with the proof of Theorem 1.1 in Section 5 following from this by integrating out the odd-location singular values. The success of this task is based on the notion of *admissible* symmetric weights, which we introduce in Section 4. There, Theorem 4.1 will give a complete classification of all admissible weights, namely, that they are exactly the Gauss, symmetric Jacobi and Cauchy weights (this is not to say that Theorem 1.1 would not hold for other ensembles, but to point out that the method of proof is limited to those cases). In the first subsection of Section 6, some inter-relationships between gap probabilities are deduced from Theorem 1.1. For n even, these have been obtained in the earlier study [8] without knowledge of Theorem 1.1. We proceed to provide the necessary working to show that this is still possible for n odd. The relative complexity serves to further highlight the advantages of a viewpoint based on singular values. We conclude in Section 7 by presenting a number of new inter-relations between the spectra of circular ensembles, which follow upon the use of a stereographic projection of the appropriate Cauchy weights to specify circular ensemble analogues of Theorem 1.1 and its various corollaries.

2 Inter-relations known from previous studies

2.1 Circular ensembles

Central to our theme is the operation of superposition, whereby eigenvalue sequences from two independent ensembles with orthogonal symmetry are superimposed, and that of decimation, meaning in the present context that only those eigenvalues with a particular parity in the ordering are observed. The best known example of these operations involves not eigenvalues on the real line as in (1.1), but rather matrix ensembles with all eigenvalues on the unit circle in the complex plane. In fact, such ensembles naturally follow from (1.1) with the Cauchy weight

$$w_\beta(x) = \frac{1}{(1+x^2)^{\beta(n-1)/2+1}}. \tag{2.1}$$

Thus, after making for each eigenvalue the change of variables

$$e^{i\theta} = \frac{1+ix}{1-ix}, \quad x = \tan(\theta/2), \tag{2.2}$$

corresponding to a stereographic mapping from the real line to the unit circle, one obtains the eigenvalue PDF on the unit circle

$$\propto |\Delta(e^{i\theta_1}, \dots, e^{i\theta_n})|^\beta, \tag{2.3}$$

referred to, in the case $\beta = 1$, as the circular orthogonal ensemble COE_n and, in the case $\beta = 2$, as the circular unitary ensemble CUE_n, see, e.g., [10, Chapter 2].

Let us superimpose two independent COE_n ensembles to obtain a new sequence of eigenangles

$$0 < \theta_1 < \theta_2 < \cdots < \theta_{2n} < 2\pi,$$

and denote it by $\mathrm{COE}_n \cup \mathrm{COE}_n$. It was conjectured by Dyson [6] and proved by Gunson [14] that

$$\mathrm{alt}(\mathrm{COE}_n \cup \mathrm{COE}_n) \overset{\mathrm{d}}{=} \mathrm{CUE}_n, \tag{2.4}$$

where the alt operation refers to the integration over alternate angles $\theta_1, \theta_3, \dots, \theta_{2n-1}$ in the region

$$\theta_{2j} < \theta_{2j+1} < \theta_{2j+2}, \quad j = 0, \dots, n-1,$$

with $\theta_0 = \theta_{2n} - 2\pi$.

The inter-relation in equation (2.4) between eigenvalue distributions implies an inter-relation between conditioned gap probabilities. These are the probabilities, denoted by $E_{n,\beta}(k; J; w_\beta)$, or alternatively by $E_{n,\beta}(k; J; \mathrm{ME}_{n,\beta}(w_\beta))$, that the matrix ensemble $\mathrm{ME}_{n,\beta}(w_\beta)$ contains exactly k eigenvalues in the interval J. Then, as a direct combinatorial consequence of (2.4), one has, see [6, 15] and cf. also (2.9), (2.11),

$$\begin{aligned} E_{n,2}(k; (-\theta, \theta); \mathrm{CUE}_n) = \sum_{j=0}^{n} &(E_{n,1}(2(k-j); (-\theta,\theta); \mathrm{COE}_n) + E_{n,1}(2(k-j)-1; (-\theta,\theta); \mathrm{COE}_n)) \\ &\times (E_{n,1}(2j; (-\theta,\theta); \mathrm{COE}_n) + E_{n,1}(2j+1; (-\theta,\theta); \mathrm{COE}_n)). \end{aligned} \tag{2.5}$$

Closely related to the determinantal structure underlying the eigenvalue PDF (2.3) for the CUE_n, together with the fact that this eigenvalue PDF is unchanged by complex conjugation, is the inter-relation, see [17],

$$|\mathrm{CUE}_n| \overset{\mathrm{d}}{=} O^+(n+1) \cup O^-(n+1). \tag{2.6}$$

As the name suggests, here $O^\pm(n+1)$ refers to the eigenangles of matrices from the classical groups of the same name, chosen with Haar measure. Eigenangles 0 and π, which appear for purely algebraic reasons, are ignored and, since orthogonal matrices have real entries, for each eigenangle $\theta \neq 0, \pi$, there is another eigenangle $-\theta$, so that we take the one within the range $0 < \theta < \pi$ only. The notation $|\cdot|$ now refers to the distribution of eigenangles in the range $0 < \theta < \pi$ union the negative of the eigenangles in the range $-\pi < \theta < 0$. Though $|\cdot|$ has no effect on $O^\pm(n+1)$, this is not the case for the CUE_n, where the eigenvalue distribution and the distribution implied by $|\mathrm{CUE}_n|$ are very different.

As shown in [8, (2.6)], the analogue of (2.6) for Hermitian matrix ensembles with unitary symmetry is (1.6). In fact, (2.6) can be deduced from (1.6) with the Cauchy weight $w_2(x) = (1+x^2)^{-n}$ upon applying the change of variables (2.2) corresponding to a stereographic projection. On the right-hand side this requires the facts that under the change of variable $x = \tan(\theta/2)$ for each eigenvalue, see [8, (2.24)–(2.28)],

$$\begin{aligned} \mathrm{chUE}_{\hat m}((1+x^2)^{-n}) &\overset{\mathrm{d}}{=} O^+(n+1), \\ \mathrm{chUE}_m(x^2(1+x^2)^{-n}) &\overset{\mathrm{d}}{=} O^-(n+1), \end{aligned} \tag{2.7}$$

and on the left-hand side this change of variables simply gives

$$|\mathrm{UE}_n((1+x^2)^{-n})| \overset{\mathrm{d}}{=} |\mathrm{CUE}_n|.$$

2.2 Hermitian ensembles

Forrester and Rains [12] considered analogues of (2.4) for ensembles of Hermitian matrices. In keeping with the above notation, let $\mathrm{OE}_n(w_1) \cup \mathrm{OE}_n(w_1)$ denote the superimposing of two sequences of eigenvalues, independently drawn from $\mathrm{OE}_n(w_1)$. Suppose the resulting eigenvalues are ordered $x_1 > x_2 > \cdots > x_{2n}$ and let $\mathrm{even}(\mathrm{OE}_n(w_1) \cup \mathrm{OE}_n(w_1))$ refer to the distribution of the even-location eigenvalues. We know from [12, pp. 186–187], see also [10, Section 6.6], that this is identically distributed to an ensemble with unitary symmetry

$$\mathrm{even}(\mathrm{OE}_n(w_1) \cup \mathrm{OE}_n(w_1)) \overset{\mathrm{d}}{=} \mathrm{UE}_n(w_2) \tag{2.8}$$

for the pairs (w_1, w_2) of weights given in Table 2 and, furthermore, up to a linear fractional transformation, these pairs of weights are *unique*. The inter-relation between ensembles (2.8) has as an immediate combinatorial consequence the inter-relation between gap probabilities

$$E_{n,2}(k; (0,s); w_2) = \sum_{j=0}^{2k} E_{n,1}(2k-j; (0,s); w_1)(E_{n,1}(j; (0,s); w_1) + E_{n,1}(j-1; (0,s); w_1)). \tag{2.9}$$

It is also fruitful to consider the superimposed and decimated ensemble $\mathrm{even}(\mathrm{OE}_n(f) \cup \mathrm{OE}_{n+1}(f))$, thus involving one ensemble with n eigenvalues and the other with $n+1$. It is shown in [12, pp. 185–186], see

Case	$w_1(x)$	$w_2(x)$	Support
Laguerre	$e^{-x/2}$	e^{-x}	$(0,\infty)$
Jacobi	$(1-x)^{(a-1)/2}$	$(1-x)^a$	$(0,1)$

Table 2. Pairs of weights satisfying (2.8); $a > -1$.

also [10, Section 6.6], that this, again, is identically distributed to an ensemble with unitary symmetry

$$\operatorname{even}(\mathrm{OE}_n(w_1) \cup \mathrm{OE}_{n+1}(w_1)) \stackrel{\mathrm{d}}{=} \mathrm{UE}_n(w_2), \tag{2.10}$$

where (w_1, w_2) is any one of the pairs (w_1, w_2) of weights given in Table 3 (note that Table 1 gives the subset of *even* weights). As for Table 2 in relation to (2.8), these pairs of weights were shown to be *unique* up to linear transformation. An immediate combinatorial consequence for gap probabilities is the inter-relation

Case	$w_1(x)$	$w_2(x)$	Support
Gauss	$e^{-x^2/2}$	e^{-x^2}	$(-\infty,\infty)$
Laguerre	$x^{(a-1)/2}e^{-x/2}$	$x^a e^{-x}$	$(0,\infty)$
Jacobi	$(1+x)^{(a-1)/2}(1-x)^{(b-1)/2}$	$(1+x)^a(1-x)^b$	$(-1,1)$
Cauchy	$(1+x^2)^{-(n+a+1)/2}$	$(1+x^2)^{-(n+a)}$	$(-\infty,\infty)$

Table 3. Pairs of weights satisfying (2.10); $a, b > -1$.

$$E_{n,2}(k; J_s; w_2) = \sum_{j=0}^{2k+1} E_{n,1}(2k+1-j; J_s; w_1)(E_{n+1,1}(j; J_s; w_1) + E_{n+1,1}(j-1; J_s; w_1)), \tag{2.11}$$

where J_s is a single interval either starting at the left boundary of support and finishing at s, or starting at s and finishing at the right boundary of support.

Remark 2.1. Although it has no direct bearing on the present study, there is a decimation relation relating $\mathrm{OE}_n(w_1)$ for the weights in Table 3 to a corresponding PDF (1.1) with $\beta = 4$, see [12, 16], which further generalises to a decimation relation reducing ensembles with $\beta = 2/(r+1)$, $r \in \mathbb{Z}^+$, to ensembles with $\beta = 2(r+1)$, see [9].

3 Joint density of the singular values of orthogonal ensembles

In this section, we assume that w_1 is an *even* weight function supported on the interval $(-\varpi, \varpi)$. By symmetry, we can establish the joint density of the singular values by restricting ourselves to the cone of increasingly ordered singular values

$$0 \le \sigma_1 \le \cdots \le \sigma_n, \tag{3.1}$$

this way parametrizing $|\mathrm{OE}_n(w_1)|$. To simplify notation and to avoid case distinctions between odd and even order n in later parts of the paper, we introduce two further sets of coordinates for this cone. Writing, as detailed in (1.3) and (1.4), $n = 2m + \mu$ and $\hat{m} = m + \mu$ with $\mu = 0, 1$, the coordinates

$$x_j = \sigma_{2j-1}, \quad j = 1, \ldots, \hat{m}, \quad y_j = \sigma_{2j}, \quad j = 1, \ldots, m, \tag{3.2}$$

satisfy the interlacing property

$$0 \le x_1 \le y_1 \le x_2 \le y_2 \le \cdots \le x_{\hat{m}} \le y_{\hat{m}} \le \varpi, \tag{3.3}$$

with formally adding, if $\mu = 1$, the value $y_{m+1} = \varpi$. With $x^{\downarrow}$ and $y^{\downarrow}$ denoting the x and y vectors with their components taken in the reverse order, so $x^{\downarrow} = (x_{\hat{m}}, x_{\hat{m}-1}, \ldots, x_1)$ and $y^{\downarrow} = (y_m, y_{m-1}, \ldots, y_1)$, we define, depending on the parity of n, the coordinates

$$(t, s) = (y^{\downarrow}, x^{\downarrow}), \quad \mu = 0, \quad (t, s) = (x^{\downarrow}, y^{\downarrow}), \quad \mu = 1, \tag{3.4}$$

satisfying the interlacing property

$$\varpi \geq t_1 \geq s_1 \geq t_2 \geq s_2 \geq \cdots \geq t_{\hat{m}} \geq s_{\hat{m}} \geq 0, \tag{3.5}$$

again formally adding the value $s_{m+1} = 0$ if $\mu = 1$. Since the mapping from $\sigma = (\sigma_1, \dots, \sigma_n)$ to either the pair of coordinates (x, y) or (t, s) is orthogonal, transforming the density between the three sets of coordinates is simply done by inserting new variable names for old ones. Note that the s variables parametrize the even-location decimated ensemble even $|\mathrm{OE}_n(w_1)|$ while the t-variables do the same for odd $|\mathrm{OE}_n(w_1)|$. We call them the even and odd singular values.

By the evenness of w_1, the joint probability density of the singular values is, supported on (3.1),

$$q(\sigma_1, \dots, \sigma_n) = n! \sum_{\epsilon \in \{\pm 1\}^n} p(\epsilon_1 \sigma_1, \dots, \epsilon_n \sigma_n) = c_{n,1} n! \cdot \prod_{k=1}^{n} w(\sigma_k) \cdot D(\sigma_1, \dots, \sigma_n)$$

with

$$D(\sigma_1, \dots, \sigma_n) = \sum_{\epsilon \in \{\pm 1\}^n} |\Delta(\epsilon_1 \sigma_1, \dots, \epsilon_n \sigma_n)|.$$

Writing $D(x; y) = D(\sigma_1, \dots, \sigma_n)$ in terms of (x, y)-coordinates, Bornemann and La Croix [3, (11)] proved in two different ways the algebraic fact that

$$D(x; y) = 2^n \cdot \Delta(x_1^2, \dots, x_{\hat{m}}^2) \cdot y_1 \cdots y_m \Delta(y_1^2, \dots, y_m^2).$$

Hence, we immediately get the following theorem.

Theorem 3.1. *Let w_1 be an even weight on $(-\varpi, \varpi)$. Then, the joint probability density of $|\mathrm{OE}_n(w_1)|$, supported on the cone (3.3), is given by*

$$q(x; y) = c_n \cdot \left(\prod_{k=1}^{\hat{m}} w_1(x_k) \cdot \Delta(x_1^2, \dots, x_{\hat{m}}^2) \right) \cdot \left(\prod_{k=1}^{m} y_k w_1(y_k) \cdot \Delta(y_1^2, \dots, y_m^2) \right) \tag{3.6}$$

with $c_n = c_{n,1} n! 2^n$.

Remark 3.2. Because of the interlacing in (3.3), this factorization does *not* reveal an independence between the x and y variables.

4 Admissible symmetric weights

We call a smooth integrable weight $w_1 : (-\varpi, \varpi) \to (0, \infty)$ *admissible* of order κ and mass

$$2\theta = \int_{-\varpi}^{\varpi} w_1(\xi) \, d\xi \tag{4.1}$$

if it satisfies the following properties:

(i) w_1 is even,

(ii) w_1 is normalized, that is, $w_1(0) = 1$,

(iii) w_1 satisfies a three-term recurrence of antiderivatives of the form

$$\int^{x} \xi^k w_1(\xi) \, d\xi = -\alpha_k x^{k-1} \phi(x) w_1(x) + \beta_k \int^{x} \xi^{k-2} w_1(\xi) \, d\xi, \quad k = 1, 2, \dots, \kappa, \tag{4.2}$$

with a smooth function $\phi : (-\varpi, \varpi) \to (0, \infty)$ and constants α_k, β_k such that $\beta_1 = 0$,

Case	Parameter	Order	$w_1(x)$	ϖ	α_k	β_k	$\phi(x)$	θ
Gauss		$\kappa < \infty$	$e^{-x^2/2}$	∞	1	$k-1$	1	$\sqrt{\frac{\pi}{2}}$
Jacobi	$a > -1$	$\kappa < \infty$	$(1-x^2)^a$	1	$\frac{1}{2a+1+k}$	$\frac{k-1}{2a+1+k}$	$1-x^2$	$\frac{\sqrt{\pi}\,\Gamma(a+1)}{2\Gamma(a+\frac{3}{2})}$
Cauchy	$a > -\frac{1}{2}$	$\kappa < 2a$	$(1+x^2)^{-a-1}$	∞	$\frac{1}{2a+1-k}$	$\frac{k-1}{2a+1-k}$	$1+x^2$	$\frac{\sqrt{\pi}\,\Gamma(a+\frac{1}{2})}{2\Gamma(a+1)}$

Table 4. Admissible symmetric weights $w_1(x)$.

(iv) w_1 vanishes at the boundary, that is,

$$\lim_{x\to\varpi} x^k w_1(x) = \lim_{x\to\varpi} x^k \phi(x) w_1(x) = 0, \quad k = 0, 1, \ldots, \kappa.$$

Table 4 lists three cases of such admissible weights; by Theorem 4.1 below, these are *all* possible cases.

By defining $\alpha_0 = 1$, $\beta_0 = 0$ and

$$\psi(x) = -\frac{1}{\phi(x)w_1(x)} \int_0^x w_1(\xi)\,d\xi, \quad -\varpi < x < \varpi,$$

the recurrence (iii) extends to the case $k = 0$ if we replace x^{-1} by $\psi(x)$. By introducing the vectors

$$\pi_\nu^n(x) = \begin{pmatrix} x^\nu \\ x^{\nu+2} \\ \vdots \\ x^{\nu+2n-2} \end{pmatrix} \in \mathbb{R}^n, \quad \nu = -1, 0, 1,$$

with the understanding that, instead of x^{-1}, the first entry of $\pi_{-1}^n(x)$ is in fact $\psi(x)$, we can write the thus extended recurrence in the compact matrix-vector form

$$\int^x w_1(\xi)\pi_\nu^n(\xi)\,d\xi = L_{n,\nu}\cdot \tilde{w}_1(x)\pi_{\nu-1}^n(x), \quad \nu = 0, 1,\ 2n+\nu \le \kappa+2, \tag{4.3}$$

$$\tilde{w}_1(x) = \phi(x)\,w_1(x), \tag{4.4}$$

with a constant *lower triangular* matrix $L_{n,\nu} \in \mathbb{R}^{n\times n}$ having the numbers $-\alpha_\nu, -\alpha_{\nu+2}, \ldots, -\alpha_{\nu+2n-2}$ along its main diagonal. In particular, there holds

$$\det L_{n,\nu} = (-1)^n A_{n,\nu}, \quad A_{n,\nu} = \prod_{k=0}^{n-1} \alpha_{2k+\nu}. \tag{4.5}$$

Since within the range of k restricted by the order κ the constants α_k and β_k given in Table 4 are strictly positive (with the exception of $\beta_1 = 0$), we have $A_{n,\nu} > 0$. We call $\tilde{w}_1 = \phi w_1$ the *companion weight* of $w_1(x)$ and observe that

$$\lim_{s\to\varpi} \tilde{w}_1(s)\psi(s) = -\theta. \tag{4.6}$$

In analogy to the results recalled in Section 2.2, we have the following *uniqueness* result.

Theorem 4.1. *Up to a rescaling of x, all possible admissible weights $w_1(x)$ are listed in Table 4. Actually, properties* (ii)–(iv) *of an admissible weight are sufficient for the conclusion to hold, that is, those properties already imply the evenness assumption* (i).

Proof. Let $w_1(x)$ be an admissible weight. Differentiating (4.2) yields

$$(x^2 - \beta_k + (k-1)\alpha_k\phi)w_1 = -\alpha_k x(\phi w_1)', \quad k = 1, 2, \ldots, \kappa. \tag{4.7}$$

Inserting $x = 0$ gives

$$\beta_k = (k-1)\alpha_k\phi(0).$$

Therefore, if $\alpha_k = 0$ for some positive integer k, we would get also that $\beta_k = 0$ and, hence, that

$$\int^{x} \xi^k w_1(\xi)\, d\xi = 0$$

in contradiction to w_1 being positive. We conclude that

$$\alpha_k \neq 0, \quad k = 1, 2, \dots, \kappa.$$

Inserting $k = 1$ into (4.7) gives the differential equation

$$(\phi w_1)' = -\frac{1}{\alpha_1} x w_1, \quad w_1(0) = 1. \tag{4.8}$$

Inserting this expression for $(\phi w_1)'$ into (4.7) and rearranging, we get

$$\frac{x^2 - \beta_k}{\alpha_k} + (k-1)\phi = \frac{x^2}{\alpha_1}, \quad k = 1, 2, \dots, \kappa.$$

Solving for ϕ gives

$$\phi(x) = \frac{1}{k-1}\frac{\alpha_k - \alpha_1}{\alpha_k \alpha_1} x^2 + \frac{1}{k-1}\frac{\beta_k}{\alpha_k} = \frac{1}{k-1}\frac{\alpha_k - \alpha_1}{\alpha_k \alpha_1} x^2 + \phi(0).$$

Since $\phi(x)$ is assumed to be *independent* of k, we get that

$$\phi(x) = \phi(0) + \tau x^2, \quad \tau = \frac{\alpha_2 - \alpha_1}{\alpha_2 \alpha_1} = \frac{1}{k-1}\frac{\alpha_k - \alpha_1}{\alpha_k \alpha_1}, \quad k = 2, 3, \dots, \kappa, \tag{4.9}$$

which can be solved for α_k yielding

$$\alpha_k = \frac{\alpha_1 \alpha_2}{\alpha_2 + (k-1)(\alpha_1 - \alpha_2)}.$$

Now, we distinguish four cases depending on whether $\phi(0)$ and τ are zero or not.

Case 1: $\phi(0) = 0$ and $\tau = 0$, that is, $\phi \equiv 0$. By (4.8) we have $x w_1 \equiv 0$, which contradicts the positivity of w_1.

Case 2: $\phi(0) = 0$ and $\tau \neq 0$. By absorbing a rescaling of the α_k into ϕ we can arrange for $\phi(x) = \pm x^2$. Now, solving the differential equation (4.8) for w_1 yields

$$w_1(x) = c x^{-2 \mp 1/\alpha_1}$$

with some constant c. For $w_1(0) = 1$ to make sense, we would need the exponent to vanish, implying that already $w_1 \equiv 1$. But such a weight would not satisfy $w_1(x) \to 0$ as $x \to \varpi$.

Case 3: $\phi(0) \neq 0$ and $\tau = 0$. By rescaling x we can arrange for $\alpha_1 = \pm 1$ and $\phi \equiv 1$. Now, solving the initial value problem (4.8) for w_1 yields

$$w_1(x) = e^{\mp x^2/2}.$$

From $w_1(x) \to 0$ as $x \to \varpi$ we get $\alpha_1 = 1$ and $\varpi = \infty$. This yields the Gauss case of Table 4.

Case 4: $\phi(0) \neq 0$ and $\alpha_1 \neq \alpha_2$. By rescaling x and absorbing a rescaling of the α_k into ϕ we can arrange for $\phi(x) = 1 \pm x^2$. Now, solving the initial value problem (4.8) for w_1 yields

$$w_1(x) = (1 \pm x^2)^{-1 \mp 1/2\alpha_1}.$$

In the case $\phi(x) = 1 - x^2$ we set $a = -1 + 1/2\alpha_1$ and get, assuring integrability,

$$w_1(x) = (1 - x^2)^a, \quad a > -1, \quad \varpi = 1, \quad \alpha_1 = \frac{1}{2a + 2},$$

which yields the symmetric Jacobi case of Table 4. In the case $\phi(x) = 1 + x^2$ we set $a = 1/2\alpha_1$ and get, once more assuring integrability,

$$w_1(x) = (1 + x^2)^{-a-1}, \quad a > -\frac{1}{2}, \quad \varpi = \infty, \quad \alpha_1 = \frac{1}{2a},$$

which finally yields the Cauchy case of Table 4 (the only case where there is a restriction of the maximum order κ that has to be checked). □

Remark 4.2. If $\phi(0) \neq 0$, (4.8) and (4.9) imply that the logarithmic derivative of $\tilde{w}_1 = \phi w_1$, namely,

$$\frac{\tilde{w}_1'}{\tilde{w}_1} = -\frac{x}{\alpha_1 \phi(x)},$$

takes the form of a ratio of a linear and a quadratic polynomial. Hence, we immediately see that $\tilde{w}_1$ must be a *classical* weight. Because of the common denominator ϕ in the logarithmic derivatives, the same conclusion holds for the weights w_1 and $w_2 = w_1 \tilde{w}_1$. Therefore, we could have finished the proof by checking properties (ii)–(iv) for each entry of a list of all classical weights.

5 Integrating out the odd and even singular values

5.1 Integrating out the odd singular values

We now prove Theorem 1.1. To begin with, we transform the joint density (3.6) to (s, t) coordinates, that is,

$$q(s; t) = c_n \cdot g_\mu(s_1, \dots, s_m) \cdot g_{1-\mu}(t_1, \dots, t_{\hat{m}})$$

with functions

$$g_\nu(z_1, \dots, z_m) = \prod_{k=1}^{m} z_k^\nu w_1(z_k) \cdot \Delta(z_m^2, \dots, z_1^2). \tag{5.1}$$

Likewise, we write $\tilde{g}_\nu$ for the same form of expression using the companion weight $\tilde{w}_1$ instead of w_1.

Now, Corollary 5.3 below shows that integrating out the odd singular values t subject to the interlacing (3.5) gives the marginal density

$$\begin{aligned} q_{\text{even}}(s_1, \dots, s_m) &= c_n \theta^\mu A_{\hat{m},1-\mu} \cdot g_\mu(s_1, \dots, s_m) \tilde{g}_\mu(s_1, \dots, s_m) \\ &= c_n \theta^\mu A_{\hat{m},1-\mu} \cdot \prod_{k=1}^{m} s_k^{2\mu} w_2(s_k) \cdot \Delta(s_m^2, \dots, s_1^2)^2, \quad \varpi \geq s_1 \geq \cdots \geq s_m \geq 0, \end{aligned} \tag{5.2}$$

of the even singular values, which is defining the associated weight function (cf. the remark in [12, p. 186])

$$w_2(s) = w_1(s)\hat{w_1}(s) = \phi(s) w_1(s)^2. \tag{5.3}$$

Since the last expression in (5.2) is easily identified as the joint density of $\text{chUE}_m(x^{2\mu} w_2)$, see (1.2), we have finally proved Theorem 1.1.

Remark 5.1. As a side product, the representation (5.2) shows that the normalization constant $c_{m,\mu}^{\text{ch}}$ of the joint density of $\text{OE}(x^{2\mu} w_1)$, if extended by symmetry to be supported on $(0, \infty)^m$, is given by

$$c_{m,\mu}^{\text{ch}} = c_{n,1} A_{\hat{m},1-\mu} \theta^\mu \frac{2^n n!}{m!}.$$

The integration is based on the following lemma and its first Corollary 5.3.

Lemma 5.2. *Let $\tilde{w}_1$ be the companion of the admissible weight w_1. Then, there holds*

$$\begin{aligned} &\int_{x_1}^{x_2} d\xi_1 \cdots \int_{x_n}^{x_{n+1}} d\xi_n \det(w_1(\xi_1)\pi_\nu^n(\xi_1) \quad \cdots \quad w_1(\xi_n)\pi_\nu^n(\xi_n)) \\ &\qquad = A_{n,\nu} \det \begin{pmatrix} \tilde{w}_1(x_1)\pi_{\nu-1}^n(x_1) & \cdots & \tilde{w}_1(x_{n+1})\pi_{\nu-1}^n(x_{n+1}) \\ 1 & \cdots & 1 \end{pmatrix}, \quad \nu = 0, 1. \end{aligned}$$

Here, all integration bounds are within $(0, \varpi)$ and, in the case of a Cauchy weight, $2n + \nu \leq \kappa + 2$.

Proof. Simplifying the notation to $\pi_\nu(x) = \pi_\nu^n(x)$, by means of (4.3), (4.4) and (4.5) we calculate

$$\begin{aligned}
&\int_{x_1}^{x_2} d\xi_1 \cdots \int_{x_n}^{x_{n+1}} d\xi_n \det(w_1(\xi_1)\pi_\nu(\xi_1) \quad \cdots \quad w_1(\xi_n)\pi_\nu(\xi_n)) \\
&\qquad = \det\Big(\int_{x_1}^{x_2} w_1(\xi_1)\pi_\nu(\xi_1)d\xi_1 \quad \cdots \quad \int_{x_n}^{x_{n+1}} w_1(\xi_n)\pi_\nu(\xi_n)d\xi_n\Big) \\
&\qquad = \underbrace{\det L_{n,\nu}}_{=(-1)^n A_{n,\nu}} \cdot \det(\tilde w_1 \pi_{\nu-1}|_{x_1}^{x_2} \quad \cdots \quad \tilde w_1 \pi_{\nu-1}|_{x_n}^{x_{n+1}}) \\
&\qquad = A_{n,\nu} \det\begin{pmatrix} \tilde w_1(x_1)\pi_{\nu-1}(x_1) & \tilde w_1\pi_{\nu-1}|_{x_1}^{x_2} & \cdots & \tilde w_1\pi_{\nu-1}|_{x_n}^{x_{n+1}} \\ 1 & 0 & \cdots & 0 \end{pmatrix} \\
&\qquad = A_{n,\nu} \det\begin{pmatrix} \tilde w_1(x_1)\pi_{\nu-1}(x_1) & \tilde w_1(x_2)\pi_{\nu-1}(x_2) & \cdots & \tilde w_1(x_{n+1})\pi_{\nu-1}(x_{n+1}) \\ 1 & 1 & \cdots & 1 \end{pmatrix}.
\end{aligned}$$

In the last step, we added the first column to the second, then the second to the third, etc. □

Corollary 5.3. *Let g_ν, $\tilde g_\nu$ be as in (5.1) and put $s_{\hat m} = 0$ if $\mu = 1$. Then, there holds*

$$\int_{s_1}^{\varpi} dt_1 \int_{s_2}^{s_1} dt_2 \cdots \int_{s_{\hat m}}^{s_{\hat m-1}} dt_{\hat m}\, g_{1-\mu}(t_1, \ldots, t_{\hat m}) = \theta^\mu A_{\hat m, 1-\mu} \cdot \tilde g_\mu(s_1, \ldots, s_m), \quad \mu = 0, 1. \tag{5.4}$$

Here, all integration bounds are within $(0, \varpi)$ and, in the case of a Cauchy weight, $n = 2m + \mu \le \kappa + 2$.

Proof. Using the notation of Lemma 5.2, we first observe that

$$g_\mu(z_1, \ldots, z_m) = \det(w_1(z_m)\pi_\mu^m(z_m) \quad \cdots \quad w_1(z_1)\pi_\mu^m(z_1)) \tag{5.5}$$

and the same for $\tilde g_\mu$ with weight $\tilde w_1$. Now, Lemma 5.2 yields, first using $\tilde w_1(s)\pi_0^m(s) \to 0$ as $s \to \varpi$, that for $\mu = 0$

$$\begin{aligned}
&\int_{s_1}^{\varpi} dt_1 \int_{s_2}^{s_1} dt_2 \cdots \int_{s_m}^{s_{m-1}} dt_m \det(w_1(t_m)\pi_1^m(t_m) \quad \cdots \quad w_1(t_1)\pi_1^m(t_1)) \\
&\qquad = A_{m,1} \det\begin{pmatrix} \tilde w_1(s_m)\pi_0^m(s_m) & \cdots & \tilde w_1(s_1)\pi_0^m(s_1) & 0 \\ 1 & \cdots & 1 & 1 \end{pmatrix} \\
&\qquad = A_{m,1} \det(\tilde w_1(s_m)\pi_0^m(s_m) \quad \cdots \quad \tilde w_1(s_1)\pi_0^m(s_1)),
\end{aligned}$$

and then, using $\pi_{-1}^{m+1}(0) = 0$ and $\tilde w_1(s)\psi(s) \to -\theta$ as $s \to \varpi$, that for $\mu = 1$

$$\begin{aligned}
&\int_{s_1}^{\varpi} dt_1 \int_{s_2}^{s_1} dt_2 \cdots \int_{0}^{s_m} dt_{m+1} \det(w_1(t_{m+1})\pi_0^{m+1}(t_{m+1}) \quad \cdots \quad w_1(t_1)\pi_0^{m+1}(t_1)) \\
&\qquad = A_{m+1,0} \cdot \det\begin{pmatrix} 0 & \tilde w_1(s_m)\pi_{-1}^{m+1}(s_m) & \cdots & \tilde w_1(s_1)\pi_{-1}^{m+1}(s_1) & \lim_{s\to\varpi} \tilde w_1(s)\pi_{-1}^{m+1}(s) \\ 1 & 1 & \cdots & 1 & 1 \end{pmatrix} \\
&\qquad = (-1)^m A_{m+1,0} \cdot \det(\tilde w_1(s_m)\pi_{-1}^{m+1}(s_m) \quad \cdots \quad \tilde w_1(s_1)\pi_{-1}^{m+1}(s_1) \quad \lim_{s\to\infty} \tilde w_1(s)\pi_{-1}^{m+1}(s)) \\
&\qquad = (-1)^m A_{m+1,0} \cdot \det\begin{pmatrix} \tilde w_1(s_m)\psi(s_m) & \cdots & \tilde w_1(s_1)\psi(s_1) & -\theta \\ \tilde w_1(s_m)\pi_1^m(s_m) & \cdots & \tilde w_1(s_1)\pi_1^m(s_1) & 0 \end{pmatrix} \\
&\qquad = \theta A_{m+1,0} \cdot \det(\tilde w_1(s_m)\pi_1^m(s_m) \quad \cdots \quad \tilde w_1(s_1)\pi_1^m(s_1)),
\end{aligned}$$

which finishes the proof. □

Remark 5.4. In the Jacobi case, the multidimensional integral (5.4) can be recognized as a variant of the Dixon–Anderson integral [1, 5], well known in the theory of the Selberg integral, and also in the theory of β-ensembles in random matrix theory, see [10, Section 4.2]. Specifically, in the statement of the Dixon–Anderson integral given in [10, (4.15)], cf. [5, (6)], that is,

$$\int_{x_1}^{x_0} d\xi_1 \cdots \int_{x_{\hat m}}^{x_{\hat m-1}} d\xi_{\hat m}\, \Delta(\xi_{\hat m},\dots,\xi_1) \prod_{j=1}^{\hat m}\prod_{k=0}^{\hat m} |\xi_j - x_k|^{a_k-1} = \frac{\prod_{i=0}^{\hat m}\Gamma(a_i)}{\Gamma(\sum_{i=0}^{\hat m} a_i)} \prod_{0\le j<k\le \hat m} (x_j - x_k)^{a_j+a_k-1},$$

valid for $x_0 > x_1 > \cdots > x_{\hat m}$ and $a_j > 0$, $j = 0,\dots,\hat m$, we can reclaim the Jacobi case of (5.4) by the substitutions of variables

$$x_0 = 1, \quad x_j = s_j^2, \quad \xi_j = t_j^2, \quad j = 1,\dots,\hat m,$$

and the choices of parameters

$$a_0 = a+1, \quad a_j = 1, \quad j = 1,\dots,m,$$

and $a_{m+1} = 1/2$, $s_{m+1} = 0$ if $\mu = 1$.

5.2 Integrating out the even singular values

The following second corollary of Lemma 5.2 will allow us to integrate out the *even* singular values from the density $q(s;t)$.

Corollary 5.5. *Let g_μ, $\tilde g_\mu$ be as in (5.1) and put $t_{m+1} = 0$ if $\mu = 0$. Then, there holds*

$$\int_{t_2}^{t_1} ds_1 \cdots \int_{t_{m+1}}^{t_m} ds_m\, g_\mu(s_1,\dots,s_m) = A_{m,\mu} \det\begin{pmatrix} \tilde w_1(t_{\hat m})\pi_{1-\mu}^{\hat m-1}(t_{\hat m}) & \cdots & \tilde w_1(t_1)\pi_{1-\mu}^{\hat m-1}(t_1) \\ \theta_{1-\mu}(t_{\hat m}) & \cdots & \theta_{1-\mu}(t_1)\end{pmatrix}$$

for $\mu = 0, 1$ with $\theta_0(x) = 1$ and

$$\theta_1(x) = \int_0^x w_1(\xi)\, d\xi.$$

Here, all integration bounds are within $(0,\varpi)$ and, in the case of a Cauchy weight, $n = 2m+\mu \le \kappa + 2$.

Proof. Using (5.5) and Lemma 5.2 we obtain

$$\begin{aligned}\int_{t_2}^{t_1} ds_1 \cdots \int_{t_{m+1}}^{t_m} ds_m\, g_\mu(s_1,\dots,s_m) &= \int_{t_2}^{t_1} ds_1 \cdots \int_{t_{m+1}}^{t_m} ds_m \det(w_1(s_m)\pi_\mu^m(s_m) \ \cdots \ w_1(s_1)\pi_\mu^m(s_1)) \\ &= A_{m,\mu}\det\begin{pmatrix}\tilde w_1(t_{m+1})\pi_{\mu-1}^m(t_{m+1}) & \cdots & \tilde w_1(t_1)\pi_{\mu-1}^m(t_1) \\ 1 & \cdots & 1\end{pmatrix},\end{aligned}$$

which is already the assertion for $\mu = 1$. For $\mu = 0$ the assertion follows from further calculating

$$\begin{aligned}&\det\begin{pmatrix}\tilde w_1(t_{m+1})\pi_{-1}^m(t_{m+1}) & \tilde w_1(t_m)\pi_{-1}^m(t_m) & \cdots & \tilde w_1(t_1)\pi_{-1}^m(t_1) \\ 1 & 1 & \cdots & 1\end{pmatrix} \\ &= \det\begin{pmatrix} 0 & \tilde w_1(t_m)\pi_{-1}^m(t_m) & \cdots & \tilde w_1(t_1)\pi_{-1}^m(t_1) \\ 1 & 1 & \cdots & 1\end{pmatrix} \\ &= (-1)^m \det(\tilde w_1(t_m)\pi_{-1}^m(t_m) \ \cdots \ \tilde w_1(t_1)\pi_{-1}^m(t_1)) \\ &= (-1)^m \det\begin{pmatrix}\tilde w_1(t_m)\psi(t_m) & \cdots & \tilde w_1(t_1)\psi(t_1) \\ \tilde w_1(t_m)\pi_1^{m-1}(t_m) & \cdots & \tilde w_1(t_1)\pi_1^{m-1}(t_1)\end{pmatrix} \\ &= \det\begin{pmatrix}\tilde w_1(t_m)\pi_1^{m-1}(t_m) & \cdots & \tilde w_1(t_1)\pi_1^{m-1}(t_1) \\ -\tilde w_1(t_m)\psi(t_m) & \cdots & -\tilde w_1(t_1)\psi(t_1)\end{pmatrix},\end{aligned}$$

which finishes the proof. □

Type	$w_1(x)$	$\tilde{w}_1(x)$	$\theta_1(x)$
Gauss	$e^{-x^2/2}$	$e^{-x^2/2}$	$\sqrt{\frac{\pi}{2}}\,\mathrm{erf}\Big(\frac{x}{\sqrt{2}}\Big)$
Jacobi	$(1-x^2)^a$	$(1-x^2)^{a+1}$	$x\cdot {}_2F_1\Big(\frac{1}{2},-a;\frac{3}{2};x^2\Big)$
Cauchy	$(1+x^2)^{-a-1}$	$(1+x^2)^{-a}$	$x\cdot {}_2F_1\Big(\frac{1}{2},a+1;\frac{3}{2};-x^2\Big)$

Table 5. Companion weights $\tilde{w}_1(x)$ and integrals $\theta_1(x)$.

Now, by means of this corollary, the marginal density of the odd singular values is given as

$$\begin{aligned} q_{\mathrm{odd}}(t_1,\dots,t_{\hat m}) &= c_n A_{m,\mu}\cdot g_{1-\mu}(t_{\hat m},\dots,t_1)\cdot \det\begin{pmatrix} \tilde{w}_1(t_{\hat m})\pi_{1-\mu}^{\hat m-1}(t_{\hat m}) & \cdots & \tilde{w}_1(t_{\hat m})\pi_{1-\mu}^{\hat m-1}(t_1) \\ \theta_{1-\mu}(t_{\hat m}) & \cdots & \theta_{1-\mu}(t_1)\end{pmatrix} \\ &= c_n A_{m,\mu}\cdot \det\begin{pmatrix} \tilde{w}_1(t_{\hat m})\pi_{1-\mu}^{\hat m-1}(t_{\hat m}) & \cdots & \tilde{w}_1(t_{\hat m})\pi_{1-\mu}^{\hat m-1}(t_1) \\ \gamma_{1-\mu}(t_{\hat m}) & \cdots & \gamma_{1-\mu}(t_1)\end{pmatrix} \\ &\quad \cdot \det\begin{pmatrix} \tilde{w}_1(t_{\hat m})\pi_{1-\mu}^{\hat m-1}(t_{\hat m}) & \cdots & \tilde{w}_1(t_{\hat m})\pi_{1-\mu}^{\hat m-1}(t_1) \\ \theta_{1-\mu}(t_{\hat m}) & \cdots & \theta_{1-\mu}(t_1)\end{pmatrix}, \quad \varpi \ge t_1 \ge \cdots \ge t_{\hat m} \ge 0, \end{aligned}$$

with

$$\gamma_\mu(x) = \tilde{w}_1(x)x^{\mu+2\hat m-2}, \qquad \theta_\mu(x) = \begin{cases} 1 & \text{if } \mu = 0, \\ \int_0^x w_1(\xi)\,d\xi & \text{if } \mu = 1. \end{cases}$$

Note that the two determinantal factors differ just in their last rows. It is this difference that prevents the expression from becoming a perfect square, which is in marked contrast with the marginal density (5.2) of the even singular values.

6 Gap probabilities

6.1 A corollary of Theorem 1.1

Theorem 1.1 has an interesting implication in terms of gap probabilities, a notion that we recalled in Section 2. Specifically, we have the following result.

Theorem 6.1. *The gap probabilities of the Gauss, symmetric Jacobi or Cauchy case of Table 1 of order $n = 2m+\mu$ satisfy*

$$E_{n,1}(2k+\mu-1;(-s,s);w_1) + E_{n,1}(2k+\mu;(-s,s);w_1) = E_{m,2}(k;(0,s^2);x^{\mu-1/2}w_2(x^{1/2})\chi_{x>0}).$$

Proof. The change of variables $x_k \mapsto \tilde{x}_k = \sqrt{x_k}$, applied to the joint density p_{ch} of the chiral ensemble $\mathrm{chUE}(x^{2\mu}w_2(x))$ yields

$$p_{\mathrm{ch}}(x_1,\dots,x_m)\,dx_1\cdots dx_m = p_{m,2}(\tilde{x}_1,\dots,\tilde{x}_m)\,d\tilde{x}_1\cdots d\tilde{x}_m,$$

where $p_{m,2}$ is the density of $\mathrm{UE}(x^{\mu-1/2}w_2(x^{1/2})\chi_{x>0})$. Hence, lifted to gap probabilities, we obtain

$$E_{m,2}(k;(0,s);\mathrm{chUE}(x^{2\mu}w_2)) = E_{m,2}(k;(0,s^2);x^{\mu-1/2}w_2(x^{1/2})\chi_{x>0}), \quad \mu = 0,1. \tag{6.1}$$

By looking at pairs of consecutive values it is easy to see that the event that exactly k values of the decimated ensemble even $|\mathrm{OE}_n(w_1)|$, $n = 2m+\mu$, are contained in $(0,s)$ is given by the union of the events that exactly $2k+\mu-1$ or that exactly $2k+\mu$ values of $|\mathrm{OE}_n(w_1)|$ are in that interval. Since these two events are *mutually* *exclusive* and since the singular values of OE_n contained in $(0, s)$ correspond to the eigenvalues in $(-s, s)$, we thus get from (1.5) and (6.1) the proof of

$$\begin{aligned} E_{n,1}(2k+\mu-1; (-s,s); w_1) + E_{n,1}(2k+\mu; (-s,s); w_1) &= E_{m,2}(k; (0,s); \mathrm{chUE}(x^{2\mu} w_2)) \\ &= E_{m,2}(k; (0,s^2); x^{\mu-1/2} w_2(x^{1/2})\chi_{x>0}), \end{aligned} \tag{6.2}$$

which finishes the proof. □

For even order ($\mu = 0$), a first proof of this theorem was given by Forrester [8, (1.14)] using generating functions, Pfaffians and Fredholm determinants. For Gaussian weights, Bornemann and La Croix [3, (40)] recently settled the odd order case by using the more elementary techniques similar to this paper.

6.2 Alternative derivation of Theorem 6.1

A natural question is to enquire if the proof of Theorem 6.1 for $\mu = 0$ given in [8] can be extended to the case $\mu = 1$. Here, we will show that the answer is positive, although as is usual for methods based on Pfaffians in the study of random matrix ensembles with $\beta = 1$, see, e.g., [10, Section 6.3.3], the number of eigenvalues being odd adds to the complexity of the calculation.

The first step is to introduce the generating function of the gap probabilities $\{E_{n,\beta}(k; J; w_\beta)\}$ according to

$$E_{n,\beta}(J; \xi; w_\beta) = \sum_{k=0}^{\infty} (1-\xi)^k E_{n,\beta}(k; J; w_\beta).$$

The generating function can be expressed as the multidimensional integral, see, e.g., [10, Proposition 8.1.2],

$$E_{n,\beta}(J; \xi; w_\beta) = \int_{-\varpi}^{\varpi} dx_1 \cdots \int_{-\varpi}^{\varpi} dx_n \prod_{j=1}^{n} (1 - \xi \chi_{x_j \in J}) \cdot p_\beta(x_1, \ldots, x_n). \tag{6.3}$$

In terms of generating functions, the assertion of Theorem 6.1 in the case $\mu = 1$ is equivalent to

$$\begin{aligned} &\left(\frac{1}{(2k)!} \frac{\partial^{2k}}{\partial \xi^{2k}} - \frac{1}{(2k+1)!} \frac{\partial^{2k+1}}{\partial \xi^{2k+1}} \right) E_{2m+1,1}((-s,s); \xi; w_1(x)) \Big|_{\xi=1} \\ &\qquad = \frac{(-1)^k}{k!} \frac{\partial^k}{\partial \xi^k} E_{m,2}((0,s^2); \xi; x^{1/2} w_2(x^{1/2})\chi_{x>0}) \Big|_{\xi=1} \end{aligned} \tag{6.4}$$

being valid for the weights in Table 1. It is this identity that we prove in the rest of the section.

By making use of Pfaffians, (6.3) for $\beta = 1$ and $w_1(x)$ even can be expressed as a determinant.

Lemma 6.2. *Let $R_j(x)$ be a polynomial of degree j for each $j = 0, 1, \ldots$, and furthermore require that $R_j(x)$ be even (odd) for j even (odd). For $w_1(x)$ even we have*

$$E_{2m+1,1}((-s,s); \xi; w_1) \propto \det Y, \tag{6.5}$$

where

$$Y = ([a_{2j-1,2k}]_{j=1,\ldots,m+1,\, k=1,\ldots,m}\ [b_{2j-1}]_{j=1,\ldots,m+1}) \tag{6.6}$$

with

$$\begin{aligned} a_{j,k} &= \frac{1}{2} \int_{-\varpi}^{\varpi} dx\, w_1(x)(1-\xi\chi_{x\in(-s,s)}) \int_{-\varpi}^{\varpi} dy\, w_1(y)(1-\xi\chi_{y\in(-s,s)}) R_{j-1}(x)\, \mathrm{sgn}(y-x) R_{k-1}(y), \\ b_j &= \frac{1}{2} \int_{-\varpi}^{\varpi} w_1(x)(1-\xi\chi_{x\in(-s,s)}) R_{j-1}(x)\, dx. \end{aligned} \tag{6.7}$$

The proportionality in (6.5) is such that the right-hand side is equal to unity when $\xi = 0$.

Proof. Let $h(x,y) = -h(y,x)$ and set

$$X = \begin{pmatrix} [h(x_j, x_k)]_{j,k=1,\dots,2m+1} & [F(x_j)]_{j=1,\dots,2m+1} \\ -[F(x_k)]_{k=1,\dots,2m+1} & 0 \end{pmatrix}. \tag{6.8}$$

It is well known, see, e.g., [10, (6.81)], that with $h(x,y) = (1/2)\operatorname{sgn}(y-x)$ and $F(x) = 1/2$ we have

$$\operatorname{Pf} X = 2^{-(m+1)} \prod_{1\le j<k\le 2m+1} \operatorname{sgn}(x_k - x_j), \tag{6.9}$$

where Pf denotes the Pfaffian. Also, it is a simple corollary of the Vandermonde determinant identity that

$$\det[R_{k-1}(x_j)]_{j,k=1,\dots,2m+1} \propto \prod_{1\le j<k\le 2m+1} (x_k - x_j). \tag{6.10}$$

Combining (6.9) and (6.10) shows that

$$\prod_{1\le j<k\le 2m+1} |x_k - x_j| \propto \det[R_{k-1}(x_j)]_{j,k=1,\dots,2m+1} \operatorname{Pf} X. \tag{6.11}$$

The significance of the decomposition (6.11) for the present purposes is that it implies a Pfaffian formula for the generating function $E_{2m+1,1}$. Specifically, substituting the definition (1.1) of the joint density $p_1(x_1,\dots,x_{2m+1})$ in (6.3) with $\beta = 1$ and then substituting (6.11) we have

$$\begin{aligned} E_{2m+1,1}((-s,s);\xi;w_1) &\propto \int_{-\varpi}^{\varpi} dx_1 \cdots \int_{-\varpi}^{\varpi} dx_{2m+1} \prod_{l=1}^{2m+1} (1 - \xi\chi_{x_l\in(-s,s)}) w_1(x_l) \det[R_{k-1}(x_j)]_{j,k=1,\dots,2m+1} \operatorname{Pf} X \\ &\propto \operatorname{Pf}\begin{pmatrix} [a_{j,k}]_{j,k=1,\dots,2m+1} & [b_j]_{j=1,\dots,2m+1} \\ -[b_k]_{k=1,\dots,2m+1} & 0 \end{pmatrix}, \end{aligned} \tag{6.12}$$

where $a_{j,k}$, b_j are given by (6.7), with the final line being a well-known identity in random matrix theory, see [4] or [10, equation (6.84)].

Finally, to obtain from this the determinant form (6.5), note that since $(1 - \xi\chi_{x\in(-s,s)})w_1(x)$ is even in x and $R_j(x)$ is even (odd) for j even (odd), we have that $a_{j,k} = 0$ when j, k have the same parity and $b_j = 0$ for j even. Thus, the nonzero entries in the Pfaffian (6.12) form a checkerboard pattern. Taking into consideration that $a_{j,k}$ is antisymmetric in the indices j, k, rearranging the rows reduces the right-hand side of (6.12) to

$$\operatorname{Pf}\begin{pmatrix} 0_{m+1} & Y \\ -Y^\top & 0_{m+1} \end{pmatrix},$$

and this in turn is equal to $\det Y$. □

At this stage the polynomials $\{R_j(x)\}$, apart from their degree and parity, are arbitrary – a judicious choice takes us closer to establishing (6.4). For this, for a given $w_2(x)$ in Table 1, introduce the family of orthonormal polynomials $\{p_j(x)\}_{j=0,\dots,n}$ such that

$$\int_{-\varpi}^{\varpi} w_2(x) p_j(x) p_k(x)\, dx = \delta_{jk}. \tag{6.13}$$

In terms of these polynomials, and the pairs of weights as implied by Table 1, choose

$$R_0(x) = 1, \quad R_{2j-1}(x) = p_{2j-1}(x), \quad R_{2j}(x) = -\frac{1}{w_1(x)}\frac{d}{dx}\Big(\frac{w_2(x)}{w_1(x)} p_{2j-1}(x)\Big), \tag{6.14}$$

where $j = 1, 2, \dots$. The latter expression is even and a polynomial of degree j since

$$\frac{1}{w_1(x)}\frac{d}{dx}\frac{w_2(x)}{w_1(x)} = -\frac{x}{\alpha_1}, \qquad \frac{w_2(x)}{w_1(x)^2} = \phi(x) = \text{even polynomial of degree } 2,$$

following from (4.8) and (5.3), which are constitutive for Table 1. We then have, for $j, k = 1, 2 \ldots,$

$$b_1 = \frac{1}{2}\int_{-\varpi}^{\varpi} w_1(x)(1-\xi\chi_{x\in(-s,s)})\,dx, \quad b_{2j+1} = \xi\frac{w_2(s)}{w_1(s)}p_{2j-1}(s), \tag{6.15}$$

and

$$a_{1,2k} = \frac{1}{2}\int_{-\varpi}^{\varpi} dx\, w_1(x)(1-\xi\chi_{x\in(-s,s)})\int_{-\varpi}^{\varpi} dy\, w_1(y)(1-\xi\chi_{y\in(-s,s)})\,\mathrm{sgn}(y-x)p_{2k-1}(y),$$

as well as

$$a_{2j+1,2k} = 2\xi\frac{w_2(s)}{w_1(s)}p_{2j-1}(s)\int_{s}^{\varpi} w_1(y)p_{2k-1}(y)\,dy - \int_{-\varpi}^{\varpi} w_2(y)(1-\xi\chi_{y\in(-s,s)})^2 p_{2j-1}(y)p_{2k-1}(y)\,dy. \tag{6.16}$$

One immediate consequence of this choice is that it allows for a simple determination of the proportionality, $1/\theta$ say, in (6.5). Thus, with $\xi = 0$ we see that $b_{2j+1} = 0$ and $a_{2j+1,2k} = -\delta_{j,k}$, where to obtain the latter use has been made of (6.13). Consequently, cf. (4.1),

$$\theta = b_1|_{\xi=0} = \frac{1}{2}\int_{-\varpi}^{\varpi} w_1(x)\,dx. \tag{6.17}$$

Let $C \in O(m)$ be a real *orthogonal* matrix and define a set $\{q_{2j-1}(x)\}_{j=1,\ldots,m}$ of polynomials by

$$\begin{pmatrix} q_1(x) \\ q_3(x) \\ \vdots \\ q_{2m-1}(x) \end{pmatrix} = C\begin{pmatrix} p_1(x) \\ p_3(x) \\ \vdots \\ p_{2m-1}(x) \end{pmatrix}. \tag{6.18}$$

If $\tilde{Y}$ is defined as for Y but with each occurrence of $p_{2j-1}(x)$ replaced by $q_{2j-1}(x)$, we get

$$\begin{pmatrix} 1 & \\ & C \end{pmatrix} Y \begin{pmatrix} C^\top & \\ & 1 \end{pmatrix} = \tilde{Y}, \quad \det\tilde{Y} = \det Y, \tag{6.19}$$

where the latter follows from $|\det C| = 1$. This allows us to make the same replacement in (6.14) and thus in (6.15) and (6.16) without having an effect on the representation (6.5) of the generating function. That this freedom leads to simplifications can be seen from the fact that $\{q_{2j-1}(x)\}$ remains an orthonormal set with respect to the inner product implied by (6.13), that is,

$$\int_{-\varpi}^{\varpi} w_2(x)q_{2j-1}(x)q_{2k-1}(x)\,dx = \delta_{jk}, \tag{6.20}$$

but can also be chosen to have an additional orthogonality as in the following lemma.

Lemma 6.3. *Define the projection kernel*

$$K(x,y) = (w_2(x)w_2(y))^{1/2}\sum_{k=1}^{m} p_{2k-1}(x)p_{2k-1}(y) \tag{6.21}$$

together with the associated integral operator

$$Kf(x) = \int_{-s}^{s} K(x,y)f(y)\,dy, \quad 0 < s < \varpi. \tag{6.22}$$

This integral operator has eigenfunctions $\{q_{2j-1}(x)\}_{j=1,\ldots,m}$ *with the structure* (6.18) *for some real orthogonal matrix C and furthermore*

$$\int_{-s}^{s} w_2(x)q_{2j-1}(x)q_{2k-1}(x)\,dx = \nu_{2j-1}(s)\delta_{jk}, \tag{6.23}$$

where $0 < \nu_{2j-1}(s) < 1$ *are the eigenvalues of K.*

This functional analytic result is essentially due to Gaudin [13], see also [10, p. 410]. The determinant of $\tilde{Y}$ can be simplified by applying the elementary column operations of replacing column k for $k = 1, \dots, n$ by column k minus

$$2\int_s^{\varpi} w_1(x) q_{2k-1}(x)\,dx$$

times column $n+1$. It is immediate that the entries in rows $2, \dots, n+1$ and columns $1, \dots, n$ are then given by

$$\begin{aligned}
\tilde{a}_{2j+1,2k} &= -\int_{-\varpi}^{\varpi} w_2(y)(1 - \xi\chi_{y\in(-s,s)})^2 q_{2j-1}(y) q_{2k-1}(y)\,dy \\
&= -\int_{-\varpi}^{\varpi} w_2(y) q_{2j-1}(y) q_{2k-1}(y)\,dy + (2\xi - \xi^2)\int_{-s}^{s} w_2(y) q_{2j-1}(y) q_{2k-1}(y)\,dy \\
&= -\delta_{jk} + (2\xi - \xi^2)\nu_{2j-1}\delta_{jk} \\
&= -\delta_{jk} + (1 - (\xi - 1)^2)\nu_{2j-1}\delta_{jk},
\end{aligned} \tag{6.24}$$

where we have used (6.13) and (6.23) to obtain the last line. The entries in row 1, column $1, \dots, n$, after first simplifying the expression for $a_{1,2k}$ in (6.15) by noting that the integral over y can be rewritten according to

$$\begin{aligned}
&\frac{1}{2}\int_{-\varpi}^{\varpi} dy\; w_1(y)(1 - \xi\chi_{y\in(-s,s)})\,\mathrm{sgn}(y - x) p_{2k-1}(y) \\
&\qquad = \xi\chi_{x\in(-s,s)}\int_s^{\varpi} w_1(t) q_{2k-1}(t)\,dt + (1 - \xi\chi_{x\in(-s,s)})\int_x^{\varpi} w_1(t) q_{2k-1}(t)\,dt,
\end{aligned}$$

now read

$$\begin{aligned}
\tilde{a}_{1,2k} &= \xi(1-\xi)\int_s^{\varpi} w_1(t) q_{2k-1}(t)\,dt \int_{-s}^{s} w_1(x)\,dx + \int_{-\varpi}^{\varpi} w_1(x)(1 - \xi\chi_{x\in(-s,s)})^2\,dx \int_x^{\varpi} w_1(t) q_{2k-1}(t)\,dt \\
&\quad - \int_s^{\varpi} w_1(t) q_{2k-1}(t)\,dt \int_{-\varpi}^{\varpi} w_1(x)(1 - \xi\chi_{x\in(-s,s)})\,dx \\
&= \int_{-\varpi}^{\varpi} w_1(x)\,dx \int_x^{\varpi} w_1(t) q_{2k-1}(t)\,dt - (1 - (\xi - 1)^2)\int_{-s}^{s} w_1(x)\,dx \int_x^{s} w_1(t) q_{2k-1}(t)\,dt.
\end{aligned} \tag{6.25}$$

The entries in the final column are unchanged by this process, and thus still have entries b_{2j-1} as specified in (6.15), with $p_{2j-1}(x)$ replaced by $q_{2j-1}(x)$.

To summarize, we have shown with (6.5), (6.17), (6.19), (6.24) and (6.25) that

$$\begin{aligned}
E_{2m+1,1}((-s,s);\xi;w_1) &= \frac{1}{\theta}\det\tilde{Y} \\
&= \begin{vmatrix} c_1^\top + c_2^\top(1 - (\xi - 1)^2) & 1 + \xi\gamma \\ -I + (1 - (\xi - 1)^2)D & \xi c_3 \end{vmatrix} \\
&= \det(I - (1 - (\xi - 1)^2)D) + \xi\begin{vmatrix} c_1^\top + c_2^\top(1 - (\xi - 1)^2) & \gamma \\ -I + (1 - (\xi - 1)^2)D & c_3 \end{vmatrix}
\end{aligned}$$

with $D = \mathrm{diag}(\nu_1(s), \nu_3(s), \dots, \nu_{2m-1}(s))$, γ a scalar and c_1, c_2, c_3 some column vectors with m entries that depend on s but not on ξ. The structure of the last formula is

$$E_{2m+1,1}((-s,s);\xi;w_1) = E(1 - (\xi - 1)^2) + \xi F(1 - (\xi - 1)^2), \tag{6.26}$$

where $F(\xi)$ is a polynomial and

$$E(\xi) = \prod_{j=1}^{m}(1 - \xi\nu_{2j-1}(s)). \tag{6.27}$$

Now, (6.26) and (6.27) are immediately amenable to the following simple lemma, which follows from a direct computation for the monomial basis $\{(1-\xi)^j\}_{j=0,1,2,\ldots}$.

Lemma 6.4. *Let $G(\xi)$ be a polynomial. Then, for $k = 0, 1, 2, \ldots$, there holds*

$$\Big(\frac{1}{(2k)!}\frac{\partial^{2k}}{\partial\xi^{2k}} - \frac{1}{(2k+1)!}\frac{\partial^{2k+1}}{\partial\xi^{2k+1}}\Big)G(1-(\xi-1)^2)\Big|_{\xi=1} = \frac{(-1)^k}{k!}\frac{\partial^k}{\partial\xi^k}G(\xi)\Big|_{\xi=1},$$

$$\Big(\frac{1}{(2k)!}\frac{\partial^{2k}}{\partial\xi^{2k}} - \frac{1}{(2k+1)!}\frac{\partial^{2k+1}}{\partial\xi^{2k+1}}\Big)\xi G(1-(\xi-1)^2)\Big|_{\xi=1} = 0.$$

An application of this lemma to (6.26) and (6.27) gives

$$\Big(\frac{1}{(2k)!}\frac{\partial^{2k}}{\partial\xi^{2k}} - \frac{1}{(2k+1)!}\frac{\partial^{2k+1}}{\partial\xi^{2k+1}}\Big)E_{2m+1,1}((-s,s);\xi;w_1) = \frac{(-1)^k}{k!}\frac{\partial^k}{\partial\xi^k}E(\xi),$$

which finally proves (6.4) by the well-known and readily established fact, see, e.g., [10, Exercises 9.6, 3], that

$$E(\xi) = E_{m,2}((0,s^2);\xi;x^{1/2}w_2(x^{1/2})\chi_{x>0}).$$

7 Circular ensembles

It was remarked in the paragraph including (2.7) that applying a stereographic projection to the eigenvalues in the appropriate Cauchy case of (1.6) gives (2.6). This transformation induces a natural definition of the decimated ensembles even $|\mathrm{COE}_n|$ and odd $|\mathrm{COE}_n|$. Now, the analogue of Theorem 1.1 allows us to characterize not only the ensemble even $|\mathrm{COE}_n|$ but also odd $|\mathrm{COE}_n|$.

Theorem 7.1. *Let μ be defined as in* (1.3), (1.4) *and, with* $\operatorname{sgn}(x) = +$ *for $x > 0$ and* $\operatorname{sgn}(x) = -$ *for $x < 0$, define* $\nu = \operatorname{sgn}(1/2-\mu)$. *Then, the circular ensembles satisfy the inter-relations*

$$\text{even}\,|\mathrm{COE}_n| \overset{\mathrm{d}}{=} O^{\nu}(n+1), \tag{7.1}$$

$$\text{odd}\,|\mathrm{COE}_n| \overset{\mathrm{d}}{=} O^{-\nu}(n+1), \tag{7.2}$$

$$|\mathrm{CUE}_n| \overset{\mathrm{d}}{=} \text{even}\,|\mathrm{COE}_n| \cup \text{odd}\,|\mathrm{COE}_n|, \tag{7.3}$$

where, in the last equation, both ensembles on the right are to be chosen independently.

Remark 7.2. The last inter-relation should be contrasted with the trivial relation

$$|\mathrm{COE}_n| \overset{\mathrm{d}}{=} \text{even}\,|\mathrm{COE}_n| \cup \text{odd}\,|\mathrm{COE}_n|$$

when both occurrences of COE_n on the right would represent one and the *same* ensemble instead of being independent.

Proof. The application of Theorem 1.1 to the Cauchy ensembles with weight (2.1) and a subsequent transformation to the circular ensembles by a stereographic projection of the eigenvalues transforms, by recalling (2.7), the inter-relation (1.5) into the first assertion (7.1).

Next, we repeat these steps with the Cauchy weight

$$w_1(x) = \frac{1}{(1+x^2)^{(n-1+a)/2+1}}, \quad a > -1, \tag{7.4}$$

which transforms by the stereographic projection into the circular Jacobi ensemble with parameter a, see [10, Section 3.9]. Though the resulting PDF becomes singular in the limit $a \to -1^+$, we know from working in the theory of the Selberg integral, see, e.g., [10, Proposition 4.1.3], that the limit effectively reduces the number of eigenvalues from n to $n-1$, by the mechanism of freezing one eigenvalue, taken to be at $\theta = \pi$. This decouples but otherwise leaves the joint distribution of the remaining eigenvalues unchanged. Noting that the freezing of an eigenvalue at $\theta = \pi$ also has the consequence of replacing the even operation by the odd operation, and after applying analogous reasoning on the right-hand side of (1.5), we deduce the second assertion (7.2).

Finally, by recalling (2.6), the last assertion (7.3) follows from (7.1) and (7.2). Alternatively, (7.3) could also have been deduced from (1.7) by the choice of the appropriate Cauchy weight, and an appropriate interpretation of the weight in the second term as just discussed. □

Remark 7.3. Interestingly, the pathway to (7.2) via the limit $a \to -1^+$ in (1.5) with weight (7.4) can also be followed in the appropriate Laguerre and Jacobi cases of (2.10) to deduce (2.8).

Analogous to the deduction of Theorem 6.1 from Theorem 1.1, as a corollary of Theorem 7.1, we get the following result.

Theorem 7.4. *With μ as in* (1.3), (1.4) *and* $\nu = \operatorname{sgn}(1/2 - \mu)$, *we have the gap probability inter-relations*

$$E_{n,1}(2k-1+\mu;(-\theta,\theta);\mathrm{COE}_n) + E_{n,1}(2k+\mu;(-\theta,\theta);\mathrm{COE}_n) = E_{m,2}(k;(0,\theta);O^{+\nu}(n+1)),$$
$$E_{n,1}(2k-\mu;(-\theta,\theta);\mathrm{COE}_n) + E_{n,1}(2k+1-\mu;(-\theta,\theta);\mathrm{COE}_n) = E_{\hat{m},2}(k;(0,\theta);O^{-\nu}(n+1)).$$

In the case n even, these inter-relations have previously been noted in [8, (3.25)], where it is remarked that it allows the gap probabilities of COE_n to be expressed as simple linear combinations of the gap probabilities of $O^{\pm}(n+1)$. One advantage of such expressions is that the ensembles $O^{\pm}(n+1)$ are determinantal point processes, see, e.g., [10, Chapter 5], allowing the corresponding gap probabilities to be expressed as Fredholm determinants, which enjoy exponentially fast numerical approximation, and thus allowing for their efficient high precision computation, see [2]. Another advantage is that the gap probabilities for determinantal point processes can be shown to obey a local limit theorem in an appropriate asymptotic regime. The inter-relations then allow for the deduction of such asymptotic behaviour for the sum of neighbouring gap probabilities in COE_n, for which no direct methods are known, see [11].

The gap probability inter-relation implied by (7.3) is exactly (2.5), even though the matrix ensemble inter-relation (2.4) used in its previous derivation is distinct from (7.3). This is a concrete example of the general fact that the family of gap-probability inter-relations specified by Theorem 6.1 or by Theorem 7.4 do not contain enough information to determine a particular matrix ensemble inter-relation, even though they are suggestive.

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