

Remarks on the $\mathcal{O}(N)$ implementation of the fast marching methodCHRISTIAN RASCH[†] AND THOMAS SATZGER[‡]*Zentrum Mathematik, Technische Universität München, 80290 München, Germany*

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The fast marching algorithm computes an approximate solution to the eikonal equation in $\mathcal{O}(N \log N)$ time, where the factor $\log N$ is due to the administration of a priority queue. Recently, [Yatziv et al. \(2006, *J. Comput. Phys.*, **212**, 393–399\)](#) have suggested using an untidy priority queue, reducing the overall complexity to $\mathcal{O}(N)$ at the price of a small error in the computed solution. In this paper we give an explicit estimate of the error introduced, which is based on a discrete comparison principle. This estimate implies, in particular, that the choice of an accuracy level that is independent of the speed function F results in the complexity bound being $\mathcal{O}(F_{\max}/F_{\min}N)$. A numerical experiment illustrates this robustness problem for large ratios F_{\max}/F_{\min} .

Keywords: fast marching method; eikonal equation; distance function; untidy priority queue.

1. Introduction

The fast marching method, introduced in [Sethian \(1996\)](#), is an efficient algorithm to compute the discrete solution to the eikonal equation $\|\nabla T(x)\|F(x) = 1$, with speed function F . The algorithm is closely related to Dijkstra's single-source shortest path algorithm and computes the grid solution, starting with the grid points adjacent to the boundary, by traversing the computational domain along increasing values of T . The total complexity of this method is $\mathcal{O}(N \log N)$, where N denotes the number of grid points. Here the factor $\log N$ comes from the administration of a priority queue.

[Tsitsiklis \(1995\)](#) independently suggested a similar algorithm. Other than [Sethian \(1996\)](#), the discretization of the eikonal equation in [Tsitsiklis \(1995\)](#) was obtained in the framework of optimal control theory. Moreover, Tsitsiklis showed that a bucket sort technique, together with a slightly different discretization, may be used to compute an approximate solution to the eikonal equation in $\mathcal{O}(N)$ time.

Another approach has recently been suggested by [Yatziv et al. \(2006\)](#): using an *untidy* priority queue in the fast marching method also reduces the complexity to optimal $\mathcal{O}(N)$. The idea is to use a bucket sort technique together with a quantization that does not distinguish between values of T within a small range. This way, while sorting the values of T becomes less expensive, an error is introduced, however, which should be of the same order as the local truncation error of the discretization. The central result of this paper is [Theorem 4.2](#), which gives a rigorous estimate of the error introduced. Before we come to the theorem, the discretization and some basic notation are introduced in [Section 2](#). Then, in [Section 3](#) we briefly recall the fast marching method and the bucket sort technique, as proposed in [Yatziv et al. \(2006\)](#). A discrete comparison principle and the error estimation are the subject of [Section 4](#). Finally, an example is given that illustrates the theoretical result.

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2. Discretization

Given a closed subset $\Gamma \subset [0, 1]^2$, we consider the eikonal equation

$$\|\nabla T(x)\| F(x) = 1 \quad (x \in [0, 1]^2 \setminus \Gamma), \quad T|_{\Gamma} = 0, \quad (2.1)$$

with continuous speed function $F: [0, 1]^2 \rightarrow \mathbb{R}$, such that there are real numbers F_{\min} and F_{\max} with

$$0 < F_{\min} \leq F(x) \leq F_{\max} \quad \forall x. \quad (2.2)$$

The computational domain $[0, 1]^2$ is endowed with a Cartesian mesh (ih, jh) ($i, j = 0, \dots, n$) of grid spacing $h = 1/n$. We partition the index set $\{0, \dots, n\}^2$ into two disjoint sets $\Omega_{\mathbb{D}}$ and $\Gamma_{\mathbb{D}}$, where $\Gamma_{\mathbb{D}}$ represents the discrete version of the set Γ . Finally, we use the scheme proposed in [Rouy & Tourin \(1992\)](#) to discretize $\|\nabla T\|$, arriving at the discrete equation

$$\begin{aligned} \max(D_{ij}^{-x}T, -D_{ij}^{+x}T, 0)^2 + \max(D_{ij}^{-y}T, -D_{ij}^{+y}T, 0)^2 &= \frac{1}{F_{ij}^2}, \quad (i, j) \in \Omega_{\mathbb{D}}, \\ T_{ij} &= 0, \quad (i, j) \in \Gamma_{\mathbb{D}}. \end{aligned} \quad (2.3)$$

Here $D_{ij}^{\pm x}T$ denotes the backward/forward finite-difference approximation of the partial derivative with respect to x , that is,

$$D_{ij}^{-x}T = \frac{T_{ij} - T_{i-1,j}}{h}, \quad -D_{ij}^{+x}T = \frac{T_{ij} - T_{i+1,j}}{h},$$

where $T_{ij} = T(ih, jh)$ and $F_{ij} = F(ih, jh)$. Furthermore, we set $T_{i,n+1} = T_{i,-1} = \infty$ and $T_{n+1,j} = T_{-1,j} = \infty$ ($i, j = 0, \dots, n$) for notational convenience.

3. The algorithm

The solution of (2.3) can be efficiently computed with the fast marching method (compare, e.g., [Sethian, 1999](#)), which is given as follows. First, all points in $\Gamma_{\mathbb{D}}$ are tagged as *known*. All points in $\Omega_{\mathbb{D}}$ that have a neighbour in $\Gamma_{\mathbb{D}}$ are tagged as *trial*. Based on the known values in $\Gamma_{\mathbb{D}}$, *trial values* are computed for all trial points using (2.3). We denote by \mathcal{N} the set of trial points, the so-called *narrow band*. Then the following iteration is performed.

- (A1) Let $(i, j) \in \mathcal{N}$ denote the point with the smallest trial value.
- (A2) Tag (i, j) as *known*, and remove it from \mathcal{N} .
- (A3) Tag all neighbours of (i, j) , which are not *known*, as *trial* (unless this has not already been done) and add them to \mathcal{N} .
- (A4) Recompute the values of T for all trial neighbours of (i, j) using (2.3) with the *known* values of T only.
- (A5) If $\mathcal{N} \neq \emptyset$ then go to (A1).

As the value of T_{ij} in (2.3) depends only on smaller values of T in the neighbouring points, the fast marching method computes a solution to (2.3).

In the narrow band \mathcal{N} the following estimate holds.

LEMMA 3.1 In every step of the fast marching method the following holds:

$$\max_{(i,j) \in \mathcal{N}} T_{ij} - \min_{(i,j) \in \mathcal{N}} T_{ij} \leq \frac{1}{F_{\min}} \cdot h.$$

Proof. Let \mathcal{N} denote the narrow band at the beginning of Step (A1) and \mathcal{N}^* denote the narrow band at the end of Step (A4) after one cycle of the fast marching method. Let $(i, j) \in \mathcal{N}$ denote the point with the smallest trial value, and let (k, l) denote some trial neighbour of (i, j) , with its trial value T_{kl} computed in Step (A4). Then

$$\frac{1}{F_{kl}^2} \geq \left(\frac{T_{kl} - T_{ij}}{h} \right)^2 \Rightarrow T_{kl} \leq \min_{(p,q) \in \mathcal{N}} T_{pq} + \frac{h}{F_{kl}}.$$

If we inductively assume the assertion to be true for \mathcal{N} then, by the last inequality, we have

$$\max_{(p,q) \in \mathcal{N}^*} T_{pq} \leq \min_{(p,q) \in \mathcal{N}} T_{pq} + \frac{h}{F_{\min}}$$

(except for the neighbours of (i, j) , all grid function values of points in \mathcal{N} remain unchanged). On the other hand, the computation of a trial value that depends on the new *known* value T_{ij} cannot yield a smaller result than T_{ij} . Thus $\min_{(p,q) \in \mathcal{N}} T_{pq} \leq \min_{(p,q) \in \mathcal{N}^*} T_{pq}$ and we are done. \square

The crucial point in the fast marching method is Step (A1). Typically, a heap-based priority queue is used in order to keep track of the grid point with the current smallest trial value T_{ij} . Here we follow the suggestion of [Yatziv et al. \(2006\)](#) to use a so-called *untidy* priority queue within the fast marching method to reach linear run-time. The idea is to quantize the trial values contained in \mathcal{N} and to pick any value that quantizes to the minimal value, instead of the minimal value.

The untidy priority queue is organized as a circular array of $n_B + 1$ ‘buckets’, denoted by B_0, \dots, B_{n_B} , to store the narrow band \mathcal{N} . These buckets are dynamic arrays implemented as singly linked lists with first in, first out (FIFO) ordering. Every bucket should contain *trial* points with *trial* values differing by at most δ , where $\delta > 0$ is a small, appropriately chosen quantity. For the choice of the point with the smallest *trial* value in Step (A1) we make no distinction between points contained in one bucket. In detail, the bucket B_r contains all trial points $(i, j) \in \mathcal{N}$ with

$$r = \lfloor T_{ij} / \delta \rfloor \bmod (n_B + 1), \quad \text{where } \delta = \frac{h}{F_{\min} \cdot n_B}. \quad (3.1)$$

By the choice of δ , it is ensured that only grid points with trial values within the range of δ belong to a common bucket, that is, $|T_{ij} - T_{kl}| < \delta$ for all $(i, j), (k, l) \in B_r$, as we have, similar to Lemma 3.1, that

$$\begin{aligned} \max_{(i,j) \in \mathcal{N}} T_{ij} - \min_{(i,j) \in \mathcal{N}} T_{ij} &< \frac{h}{F_{\min}} + \delta = \delta \cdot (n_B + 1) \\ \Rightarrow \left\lfloor \max_{(i,j) \in \mathcal{N}} T_{ij} / \delta \right\rfloor - \left\lfloor \min_{(i,j) \in \mathcal{N}} T_{ij} / \delta \right\rfloor &< n_B + 1, \end{aligned}$$

when using the untidy priority queue. On the other hand, the modulus operation in (3.1) has the effect that buckets that are emptied during the algorithm are refilled in subsequent cycles of the fast marching method for memory efficiency reasons. During the computation we have to keep track of the bucket B_s , which holds the grid points with the smallest trial values. The untidy queue supports the following two operations.

- *Insertion of some point (i, j) .* To insert some point (i, j) we compute r by (3.1) and attach (i, j) at the end of the bucket B_r .
- *Deletion of the point (i, j) with the approximately minimal trial value.* The index s should hold the number of the bucket with the smallest trial value. If B_s is empty then we search cyclically for the next nonempty bucket $B_{s'}$, passing cyclically from bucket to bucket and substituting s by $\lfloor s + 1 \rfloor \bmod (n_B + 1)$. If all buckets are empty then the method terminates. Otherwise, we return the first element (i, j) in B_s and remove it from B_s .

Since the recomputation of some trial value T_{kl} in Step (A4) can yield a smaller value, it might become necessary to move (k, l) to another bucket. In our approach we decided *to insert a grid point once more* if it gets a new trial value \hat{T}_{kl} such that the corresponding bucket number changes. Additionally, if we encounter some already *known* point in the delete operation then we simply skip the point and pick the next one. This approach may have the effect that a grid point appears up to four times in the untidy priority queue.¹ However, in this way we avoid the costly removal of trial points from a bucket if they would have to be transferred to another one.

The complexity of the insert operation is obviously $\mathcal{O}(1)$. However, the computational cost of getting the next minimal element in the queue may not be constant in a single step as it incorporates the search for a nonempty bucket. Nevertheless, we obtain linear complexity in total, as the following lemma shows.

LEMMA 3.2 The complexity of the modified fast marching method with n_B buckets is $\mathcal{O}(N + n \cdot n_B)$, where $N = |\Omega_D|$.

Proof. In every cycle (A1)–(A5) of the fast marching method one grid point becomes *known*, and after N cycles the method terminates. Steps (A3) and (A4) require $\mathcal{O}(1)$ operations. But possibly several buckets have to be passed in the delete operation in Step (A2). However, the solution of (2.3) computed by the modified method is bounded from above by $M = 2/F_{\min}$. Let us imagine for simplicity that we use a linear array B_0, B_1, B_2, \dots of buckets for our untidy priority queue, disregarding the modulus operation in (3.1). Then we could cope with $\lfloor M/\delta \rfloor$ buckets. Hence, also with a circular array, at most $M/\delta = 2/h \cdot n_B = \mathcal{O}(n \cdot n_B)$ buckets have to be traversed in the delete operation during the whole algorithm. \square

4. Estimates

Let $N_{ij}(T)$ represent the approximation of $\|\nabla T(ih, jh)\|$ as given in formula (2.3). Then the following comparison principle holds.

LEMMA 4.1 Let S_{ij} and T_{ij} denote two grid functions with $N_{ij}(S)F_{ij} \leq 1$ and $N_{ij}(T)F_{ij} \geq 1$ for all $(i, j) \in \Omega_D$. Then

$$\max_{(i,j) \in \Omega_D} (S_{ij} - T_{ij})^+ \leq \max_{(i,j) \in \Gamma_D} (S_{ij} - T_{ij})^+.$$

Proof. Assume that, for some inner grid point $(i, j) \in \Omega_D$, we have $S_{ij} - T_{ij} = \max_{(i,j) \in \Omega_D} (S_{ij} - T_{ij}) = \rho > 0$ and that additionally (i, j) is the grid point with the minimal value of T_{ij} among all grid points

¹ In general, we observe that every point is inserted two times on average.

in Ω_D realizing that maximum. The solution X of the discrete equation

$$\max(X - T_{i-1,j}, X - T_{i+1,j}, 0)^2 + \max(X - T_{i,j-1}, X - T_{i,j+1}, 0)^2 = \frac{h^2}{F_{ij}^2} \quad (4.1)$$

at (i, j) can be written as²

$$X = \min_{s+t=1, s,t \geq 0} \left\{ s \cdot \min(T_{i-1,j}, T_{i+1,j}) + t \cdot \min(T_{i,j-1}, T_{i,j+1}) + \frac{h}{F_{ij}} \sqrt{s^2 + t^2} \right\}.$$

Hence, by the assumption $N_{ij}(T)F_{ij} \geq 1$, we have $T_{ij} \geq X$, and there are numbers $s, t \geq 0$ with $s + t = 1$ and $\mu, \nu \in \{-1, 1\}$ such that

$$T_{ij} \geq s \cdot T_{i+\mu,j} + t \cdot T_{i,j+\nu} + \frac{h}{F_{ij}} \sqrt{s^2 + t^2}. \quad (4.2)$$

Similarly, we deduce by $N_{ij}(S)F_{ij} \leq 1$ that

$$S_{ij} \leq s \cdot S_{i+\mu,j} + t \cdot S_{i,j+\nu} + \frac{h}{F_{ij}} \sqrt{s^2 + t^2}.$$

Taking the difference of the last two equations yields that

$$\rho = S_{ij} - T_{ij} \leq s \cdot (S_{i+\mu,j} - T_{i+\mu,j}) + t \cdot (S_{i,j+\nu} - T_{i,j+\nu}) \leq \rho.$$

By the choice of ρ , we would have $S_{i+\mu,j} - T_{i+\mu,j} = \rho$ if $s > 0$ and $S_{i,j+\nu} - T_{i,j+\nu} = \rho$ if $t > 0$. Together with (4.2), we deduce that T_{ij} has a neighbour T_{kl} with $T_{kl} < T_{ij}$ and $S_{kl} - T_{kl} = \rho$. By our assumption, T_{kl} has to be a boundary point. Thus the lemma is proved. \square

With the help of the discrete comparison principle, we are now able to prove an estimate on the error caused by the inexact minimization in the untidy priority queue.

THEOREM 4.2 Let \hat{T} denote the solution computed with the modified fast marching method, and let T denote the exact solution of (2.3). Then, for all $(i, j) \in \Omega_D$, we have

$$0 \leq \frac{\hat{T}_{ij} - T_{ij}}{|T_{ij}|} \leq \sqrt{2} \cdot \frac{F_{\max}}{h} \cdot \delta = \sqrt{2} \cdot \frac{F_{\max}}{F_{\min}} \cdot \frac{1}{n_B},$$

where $n_B + 1$ denotes the number of buckets and $\delta = \frac{h}{F_{\min}^{n_B}}$ is the increment.

Proof. We analyse the error due to the untidy priority queue. When some value \hat{T}_{ij} becomes *known* then there might be a smaller value \hat{T}_{kl} in the same bucket that would have been accepted before \hat{T}_{ij} if an exact priority queue had been used, with $\hat{T}_{kl} \leq \hat{T}_{ij} < \hat{T}_{kl} + \delta$. Let X denote the solution of

$$\max(X - \hat{T}_{i-1,j}, X - \hat{T}_{i+1,j}, 0)^2 + \max(X - \hat{T}_{i,j-1}, X - \hat{T}_{i,j+1}, 0)^2 = \frac{h^2}{F_{ij}^2}.$$

² In Bornemann & Rasch (2006) a variational discretization of the eikonal equation leads directly to this formula. As easily verified by solving a scalar extreme value problem, X satisfies (4.1).

Of course, we have $X \leq \hat{T}_{ij}$. Since, as we have seen, all values \hat{T}_{kl} that are accepted after \hat{T}_{ij} fulfil $\hat{T}_{kl} > \hat{T}_{ij} - \delta$, we also have $X > \hat{T}_{ij} - \delta$. Hence

$$\begin{aligned} \frac{h^2}{F_{ij}^2} &\leq \max(\hat{T}_{ij} - \hat{T}_{i-1,j}, \hat{T}_{ij} - \hat{T}_{i+1,j}, 0)^2 + \max(\hat{T}_{ij} - \hat{T}_{i,j-1}, \hat{T}_{ij} - \hat{T}_{i,j+1}, 0)^2 \\ &< [\max(X - \hat{T}_{i-1,j}, X - \hat{T}_{i+1,j}, 0) + \delta]^2 + [\max(X - \hat{T}_{i,j-1}, X - \hat{T}_{i,j+1}, 0) + \delta]^2 \\ &\leq \left(\frac{h}{F_{ij}} + \sqrt{2} \cdot \delta \right)^2 \leq \frac{h^2}{F_{ij}^2} \left(1 + \sqrt{2} \cdot \delta \cdot \frac{F_{\max}}{h} \right)^2. \end{aligned}$$

Thus $N_{ij}(\hat{T})F_{ij} \geq 1$ and, as $N_{ij}(T)F_{ij} = 1$, we obtain from Lemma 4.1 that $T_{ij} \leq \hat{T}_{ij}$ for all (i, j) . On the other hand, we have $\theta \cdot N_{ij}(\hat{T})F_{ij} = N_{ij}(\theta \cdot \hat{T})F_{ij} \leq 1$ with $\theta = (1 + \sqrt{2}\delta F_{\max}/h)^{-1}$, $0 < \theta < 1$. Once again, Lemma 4.1 yields that

$$0 \leq \hat{T}_{ij} - T_{ij} \leq (1 - \theta) \cdot \hat{T}_{ij} + \max_{(k,l) \in \Omega_D} (\theta \hat{T}_{kl} - T_{kl})^+ \leq (1 - \theta) \cdot \hat{T}_{ij}.$$

Hence we get that \hat{T}_{ij} is bounded by $\theta^{-1} \cdot T_{ij}$. \square

REMARKS 4.3

1. Of course, the result in Theorem 4.2 can be generalized to arbitrary space dimensions. In the estimate on the relative error the factor $\sqrt{2}$ has then to be substituted by the square root of the space dimension.
2. As Lemma 3.2 and Theorem 4.2 show, an increase in the number of buckets causes a lower error at the price of a higher complexity. Moreover, we expect a total error of at least $\mathcal{O}(h)$ due to the discretization of the eikonal equation and a complexity of at least $\mathcal{O}(N)$ as N is the number of grid points. Hence the suitable choice for n_B is $n_B = \mathcal{O}(n) = \mathcal{O}(1/h)$. A choice of $n_B = F_{\max}/F_{\min} \cdot n$ buckets would yield an error bound that is independent of the speed ratio F_{\max}/F_{\min} but a total complexity of $\mathcal{O}(F_{\max}/F_{\min} \cdot N)$.
3. Zhao (2005) proposes an iterative method to solve (2.3) in $\mathcal{O}(N)$ time. However, the complexity of this fast sweeping method is affected by the maximum curvature of a characteristic curve $\max_{x \in \Omega} (\|\nabla F(x)\|/F(x))$, as Zhao shows. We expect advantages using the modified fast marching method when the ratio F_{\max}/F_{\min} is moderate but F varies a lot.
4. The untidy priority queue can be used to modify the fast marching method on triangulations (Kimmel & Sethian, 1998), the method for the solution of the eikonal equation on parametric manifolds (Spira & Kimmel, 2004) or the ordered upwind method for more general Hamilton–Jacobi equations given in Sethian & Vladimirovsky (2001) to reduce the run-time to $\mathcal{O}(N)$.

5. Examples

The first example concerns the error estimation asserted in Theorem 4.2. We used the speed function $F(x) = (1 + \frac{r-1}{2}) + \frac{r-1}{2} \sin(2\pi|x|)$ with $r = F_{\max}/F_{\min}$ on $\Omega = [0, 1]^2$ and the boundary value $T(1, 0) = 0$. Figure 1 illustrates the linear dependence of the relative error $(\hat{T}_{ij} - T_{ij})/\hat{T}_{ij}$ on the ratio F_{\max}/F_{\min} for different numbers of buckets.

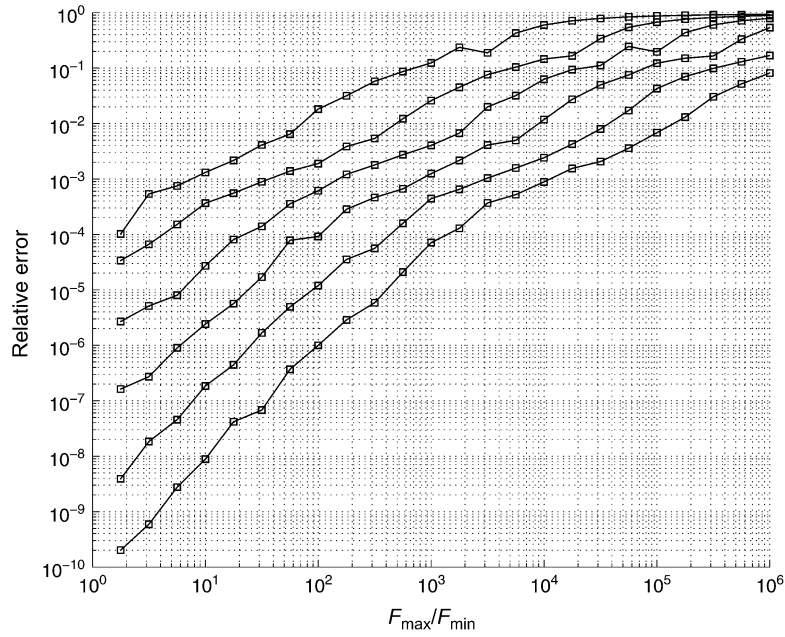


FIG. 1. Relative error versus the ratio F_{\max}/F_{\min} for different numbers of buckets (from top to bottom we used 2, 8, 32, 128, 512 and 2048 buckets).

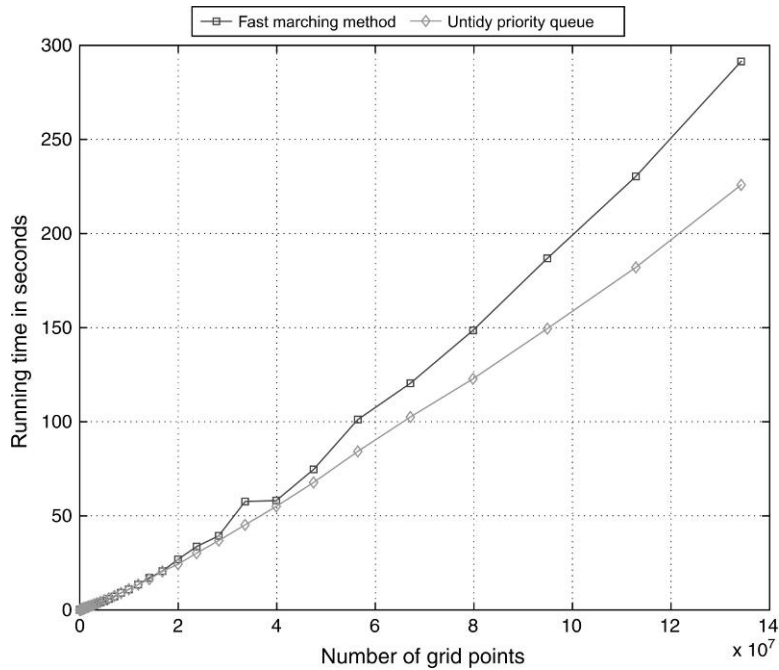


FIG. 2. Run-time of the algorithm with an untidy priority queue compared to the original fast marching method.

The second example compares the run-time of the algorithm with an untidy priority queue to the original fast marching method. We computed T_{ij} on $\Omega = [0, 1]^2$ with the speed function $1/F$ sampled from a $[0, 1]$ -uniform distributed random variable. Figure 2 shows an asymptotically linear run-time of the modified method and a nonlinear behaviour of the original method.

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