

Set oriented construction of globally optimal controllers

Mengenorientierte Konstruktion global optimaler Regler

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In den letzten Jahren wurden Techniken zur Konstruktion optimaler Regler entwickelt, die auf einer stückweise konstanten Approximation der Wertefunktion des zugrundeliegenden Optimalsteuerungsproblems beruhen. Sie kombinieren Ideen der mengenorientierten Numerik mit Algorithmen zur Berechnung kürzester Wege aus der Graphentheorie. Der Ansatz ist insbesondere gut geeignet für Probleme mit hochgradig irregulärer Wertefunktion, komplexen Zustandsbeschränkungen und ermöglicht auf natürliche Weise die Behandlung von hybriden Systemen. In diesem Beitrag geben wir einen Überblick dieses Ansatzes und illustrieren ihn anhand einiger numerischer Beispiele.

Recently, techniques for the design of optimal controllers have been developed which are based on a piecewise constant approximation of the value function of the underlying optimal control problem. They combine ideas from set oriented numerics with shortest path algorithms from graph theory. The approach is particularly well suited for problems with highly irregular value function, complicated state constraints and naturally handles hybrid systems. In this contribution, we give an overview of the approach and illustrate it by several numerical examples.

Schlagwörter: stabilisierender Regler, dynamische Programmierung, mengenorientierte Numerik, Kürzeste-Wege-Algorithmus, dynamisches Spiel, hybrides System

Keywords: stabilizing controller, dynamic programming, set oriented numerics, shortest path algorithm, dynamic game, hybrid system

1 Introduction

An elegant way to construct a globally stabilizing controller for a (nonlinear) control system is via Bellman's *optimality principle*. This is a fixed point equation which – together with a suitable boundary condition – characterizes the *value function* of the system.¹ From the value function, the associated optimally stabilizing controller can be computed by repeatedly solving a finite-dimensional optimization problem. The value function will then act as a Lyapunov function for the closed loop system.

In the case of linear dynamics and a quadratic (instantaneous) cost function, an explicit formula for V as well

¹ The value function $V(x)$ of a state x of the system is the minimal cost accumulated along any controlled trajectory starting at x . The associated instantaneous cost may be given in the problem description or may have to be suitably defined.

as the controller can be derived by solving an associated *Riccati equation*. In the nonlinear case, one typically has to resort to a numerical approximation of V . In this case, one needs to project the Bellman equation onto a finite-dimensional approximation space and solve the resulting (discrete) fixed point equation. Typically, piecewise (multi)linear approximation spaces are employed [1, 4]. Recently, however, an approximation space consisting of piecewise constant functions has been used for this purpose [12, 6, 5, 7, 8, 9]. The resulting discrete Bellman equation can be interpreted as a shortest path problem on a suitably defined graph and thus can readily be solved by fast shortest path algorithms.

The approach is particularly well suited for problems with highly irregular value function, complicated state constraints and naturally handles hybrid systems. Its extension to perturbed systems conceptually enables the treatment of discrete event systems.

2 Setting

Problem formulation. Our goal is to globally and optimally stabilize a given subset T of the state space X of the perturbed discrete-time control system

$$x_{k+1} = f(x_k, u_k, w_k), \quad k = 0, 1, \dots, \quad (1)$$

$x_k \in X$, subject to the continuous instantaneous cost

$$g(x_k, u_k) \geq 0.$$

To this end, we would like to construct an approximate *optimal feedback* $u(x_k)$ such that T is an asymptotically stable set for the resulting closed loop system

$$x_{k+1} = f(x_k, u(x_k), w_k), \quad k = 0, 1, \dots \quad (2)$$

for *any* sequence (w_k) of perturbations.

System classes, extensions. The class of systems which can be modelled by (1) is rather large, it includes

- sampled-data continuous time systems: $X \subset \mathbb{R}^n$ compact, f is the time- T -map of the control flow and $g(x_k, u_k)$ typically contains integral expressions along the continuous time solution over a sampling interval;
- certain types of hybrid systems: $X = Y \times D$, where $Y \subset \mathbb{R}^n$ compact and D is finite, cf. section 6 for details;
- discrete event systems: here f is defined as a (generalized) Poincaré map.

The controls u_k and perturbations w_k have to be chosen from compact metric spaces U and W , respectively (which may in particular be discrete).

3 Basic construction

The construction of the feedback controller will be based on the optimality principle. To this end we compute an approximation to the optimal value function of the problem which is piecewise constant on the elements of a partition of X . This type of discretization leads to a *discrete optimality principle*, i.e. a shortest path problem on a directed graph. In order to stress the basic ideas in this section we consider problems without perturbations,

$$x_{k+1} = f(x_k, u_k), \quad k = 0, 1, \dots, \quad (3)$$

and the simple target set $T = \{0\}$.

3.1 The optimality principle

We assume the map $f : X \times U \rightarrow \mathbb{R}^d$ to be continuous on some compact subset $X \times U \subset \mathbb{R}^d \times \mathbb{R}^m$ which contains the origin, $f(0, 0) = 0$ and $\inf_{u \in U} g(x, u) > 0$ for all $x \neq 0$.

For a given initial state $x \in X$ and a given control sequence $\mathbf{u} = (u_0, u_1, \dots) \in U^{\mathbb{N}}$ there is a unique associated trajectory $(x_k(x, \mathbf{u}))_{k \in \mathbb{N}}$ of (3). Let $\mathcal{U}(x) = \{\mathbf{u} \in$

$U^{\mathbb{N}} : x_k(x, \mathbf{u}) \rightarrow 0 \text{ as } k \rightarrow \infty\}$ denote the set of asymptotically controlling sequences for $x \in X$ and $S = \{x \in X : \mathcal{U}(x) \neq \emptyset\}$ the *stabilizable subset* $S \subset X$. The total cost along a controlled trajectory is given by

$$J(x, \mathbf{u}) = \sum_{k=0}^{\infty} g(x_k(x, \mathbf{u}), u_k) \in [0, \infty]. \quad (4)$$

The (*optimal*) *value function* $V(x) = \inf_{\mathbf{u} \in \mathcal{U}(x)} J(x, \mathbf{u})$ satisfies the *optimality principle*

$$V(x) = \inf_{u \in U} \{g(x, u) + V(f(x, u))\} \quad (5)$$

in all states x for which $V(x)$ is finite. The operator

$$L[v](x) := \inf_{u \in U} \{g(x, u) + V(f(x, u))\},$$

acting on real valued functions on X , is called the *dynamic programming operator* and V is the unique fixed point of the equation $v = L[v]$ satisfying the boundary condition $V(0) = 0$.

Using the optimality principle, an optimal stabilizing feedback is given by

$$u(x) = \operatorname{argmin}_{u \in U} \{g(x, u) + V(f(x, u))\}, \quad (6)$$

whenever this minimum exists. The key property in order to prove asymptotic stability of the closed loop system is the fact that by the (obvious) inequality

$$V(x) \geq g(x, u(x)) + V(f(x, u(x))) \quad (7)$$

the optimal value function is a Lyapunov function of the closed loop system, provided V is finite, positive definite and proper.²

3.2 Discretization

We are going to approximate V by functions which are piecewise constant. This approach is motivated on the one hand by the fact that the resulting discrete problem can be solved by fast shortest path algorithms and on the other hand by the observation that – via a suitable generalization to perturbed systems, cf. Section 4 – the resulting feedback is piecewise constant, too, and can be computed offline.

Let \mathcal{P} be a partition of the state space X , i.e. a collection of pairwise disjoint subsets of X whose union covers X . For a state $x \in X$ we let $\rho(x) \in \mathcal{P}$ denote the partition element which contains x . In order to simplify notation, in the sequel we identify any subset $\{P_1, \dots, P_k\} \subseteq \mathcal{P}$ with the corresponding subset $\cup_{i=1, \dots, k} P_i \subseteq X$.

Let $\mathbb{R}^{\mathcal{P}}$ be the subspace of the space \mathbb{R}^X of all real valued functions on X which are piecewise constant on the elements of the partition \mathcal{P} . The map

$$\varphi[v](x) := \inf_{x' \in \rho(x)} v(x')$$

² These properties of V can be ensured by suitable asymptotic controllability properties and bounds on g .

is a projection from \mathbb{R}^X onto \mathbb{R}^P . Using this projection, we can define the *discretized dynamic programming operator* $L_{\mathcal{P}} : \mathbb{R}^P \rightarrow \mathbb{R}^P$ by

$$L_{\mathcal{P}} := \varphi \circ L.$$

Under the boundary condition $V_{\mathcal{P}}(x) = 0$ for all $x \in \rho(0)$ this operator has a unique fixed point $V_{\mathcal{P}}$ – the *approximate (optimal) value function*.

The discrete optimality principle. Since $V_{\mathcal{P}}$ is constant on each partition element $P \in \mathcal{P}$, we write $V_{\mathcal{P}}(P)$ for the value of $V_{\mathcal{P}}$ on P . Using this notation, one can show [9] that the fixed point equation $V_{\mathcal{P}} = L_{\mathcal{P}}[V_{\mathcal{P}}]$ is equivalent to the *discrete optimality principle*

$$V_{\mathcal{P}}(P) = \min_{P' \in \mathcal{F}(P)} \{\mathcal{G}(P, P') + V_{\mathcal{P}}(P')\}, \quad (8)$$

where the map \mathcal{F} is given by

$$\begin{aligned} \mathcal{F}(P) &= \rho(f(P, U)) \\ &= \{P' \in \mathcal{P} : P' \cap f(P, U) \neq \emptyset\} \end{aligned} \quad (9)$$

and the cost function \mathcal{G} by

$$\mathcal{G}(P, P') = \inf_{u \in U} \{g(x, u) \mid x \in P, f(x, u) \in P'\}. \quad (10)$$

Note that the value $V_{\mathcal{P}}(P)$ is the length of the shortest path from P to $\rho(0)$ in the weighted directed graph (\mathcal{P}, E) , where the set of edges is defined by

$$E = \{(P, P') : P' \in \mathcal{F}(P)\}$$

and the edge (P, P') is weighted by $\mathcal{G}(P, P')$, cf. Figure 1. As such, it can be computed by, e.g., Dijkstra's algorithm, cf. [12].

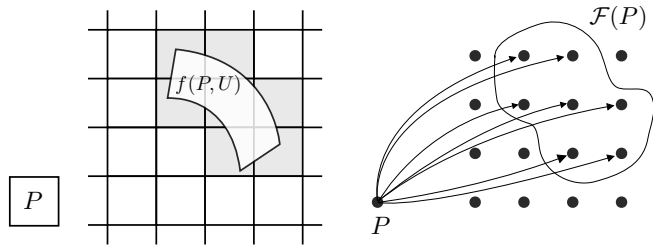


Bild 1: Partition of phase space, image of an element (left) and corresponding edges in the induced graph (right).

Properties of $V_{\mathcal{P}}$. From the definition of the projection φ and by an induction argument on the elements of the partition one can conclude that

$$V_{\mathcal{P}}(x) \leq V(x) \quad (11)$$

for any $x \in S$ and every partition \mathcal{P} , i.e. the approximate value function $V_{\mathcal{P}}$ yields a *lower bound* on the true value function. As a result, the residual

$$e_{\mathcal{P}} = L_{\mathcal{P}}[V_{\mathcal{P}}] - V_{\mathcal{P}}$$

is an efficient pointwise a posteriori estimator for the error $e = V - V_{\mathcal{P}}$, i.e. one has $e_{\mathcal{P}} \leq e$ pointwise on S [6].

3.3 Example: a simple 1D system

Consider the system

$$x_{k+1} = x_k + (1 - a)u_k x_k, \quad k = 0, 1, \dots, \quad (12)$$

where $x_k \in X = [0, 1]$, $u_k \in U = [-1, 1]$ and $a \in (0, 1)$ is a fixed parameter. Let

$$g(x, u) = (1 - a)x,$$

such that the optimal control policy is to steer to the origin as fast as possible, i.e. for every x , the optimal sequence of controls is $(-1, -1, \dots)$. This yields $V(x) = x$ as the optimal value function.

For the following computations we consider $a = 0.8$ and use a partition of 64 equally sized subintervals of $[0, 1]$. The weights (10) are approximated by minimizing over 100 equally spaced test points in each subinterval and 10 equally spaced points in U .

Figure 2 shows V and the approximate optimal value function $V_{\mathcal{P}}$.

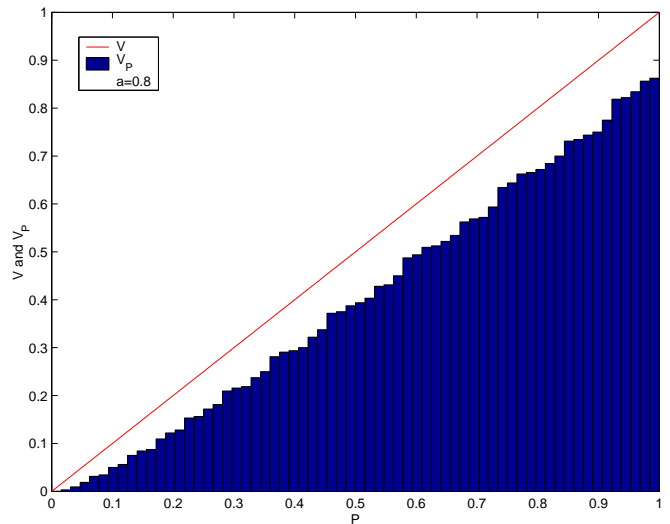


Bild 2: True and approximate optimal value function for the simple example, 64 subintervals, $a = 0.8$.

4 Construction including perturbations

In many cases, the discretization described above is not very efficient in the sense that rather fine partitions are required in order to obtain a stabilizing controller. A more efficient discretization results by employing ideas from dynamic game theory. In this section we briefly sketch how our approach can be extended to dynamic games, for details we refer to [9], and in the following Section 5 we show how this extension can be used in order to obtain stabilizing controllers on coarser partitions.

4.1 Dynamic games

A *dynamic game* is a map $F : X \times U \times W \rightarrow X$, where $X \subset \mathbb{R}^d$, $U \subset \mathbb{R}^m$, $W \subset \mathbb{R}^\ell$ are compact sets, together with a cost function $G : X \times U \times W \rightarrow [0, \infty)$. For a given initial state $x \in X$, a given control sequence $\mathbf{u} = (u_k)_{k \in \mathbb{N}} \in U^{\mathbb{N}}$ and a given perturbation sequence $\mathbf{w} = (w_k)_{k \in \mathbb{N}} \in W^{\mathbb{N}}$, the associated trajectory of the game is given by the sequence $(x_k(x, \mathbf{u}, \mathbf{w}))_{k \in \mathbb{N}}$ with

$$x_{k+1} = F(x_k, u_k, w_k), \quad k = 0, 1, 2, \dots$$

Specifying a target set $T \subset X$, the total cost accumulated along a trajectory is

$$J(x, \mathbf{u}, \mathbf{w}) = \sum_{k=0}^k G(x_k(x, \mathbf{u}, \mathbf{w}), u_k, w_k),$$

with $k = k(T, x, \mathbf{u}, \mathbf{w}) := \inf\{k \geq 0 \mid x_k(x, \mathbf{u}, \mathbf{w}) \in T\}$. Here we are interested in what is called the *upper value function* of the game, which is defined by

$$V(x) = \sup_{\beta \in \mathcal{B}} \inf_{\mathbf{u} \in U^{\mathbb{N}}} J(x, \mathbf{u}, \beta(\mathbf{u})). \quad (13)$$

Here, \mathcal{B} denotes the set of all nonanticipating strategies $\beta : U^{\mathbb{N}} \rightarrow W^{\mathbb{N}}$, i.e. all strategies β satisfying

$$u_k = u'_k \quad \forall k \leq K \quad \Rightarrow \quad \beta(\mathbf{u})_k = \beta(\mathbf{u}')_k \quad \forall k \leq K$$

for any two control sequences $\mathbf{u} = (u_k)_k$, $\mathbf{u}' = (u'_k)_k \in U^{\mathbb{N}}$. By standard dynamic programming arguments [2] one sees that this function is the unique solution to the optimality principle

$$V(x) = \inf_{u \in U} \sup_{w \in W} \{G(x, u, w) + V(F(x, u, w))\} \quad (14)$$

for $x \notin T$ together with the boundary condition $V|_T \equiv 0$. If G does not depend on w , then this equation can be written as

$$V(x) = \inf_{u \in U} \left\{ G(x, u) + \sup_{x' \in F(x, u, W)} V(x') \right\}. \quad (15)$$

Note that in this equation it is sufficient to know the set valued image $F(x, u, W)$. The discretization described in the following section will be based on this observation.

4.2 Discretization of the dynamic game

We employ the same approach as in Section 3.2 in order to discretize (14). Note that the setting in Section 3.2 can be seen as a special case of the more general situation here using $W = \{0\}$ and $F(x, u, w) = f(x, u)$. In our perturbed setup, one can show [9] that the corresponding discrete optimality principle is given by

$$V_{\mathcal{P}}(P) = \inf_{\mathcal{N} \in \mathcal{F}(P)} \left\{ \mathcal{G}(P, \mathcal{N}) + \sup_{P' \in \mathcal{N}} V_{\mathcal{P}}(P') \right\} \quad (16)$$

for $P \cap T = \emptyset$, with boundary condition $V_{\mathcal{P}}(P) = 0$ for $P \cap T \neq \emptyset$, where

$$\mathcal{F}(P) = \{\rho(F(x, u, W)) : (x, u) \in P \times U\}. \quad (17)$$

and

$$\mathcal{G}(P, \mathcal{N}) = \inf \left\{ G(x, u) : \begin{array}{l} (x, u) \in P \times U, \\ \rho(F(x, u, W)) = \mathcal{N} \end{array} \right\}.$$

Note the difference of $\mathcal{F}(P)$ compared to (10): while in (10) $\mathcal{F}(P)$ was a subset of \mathcal{P} , in (17) $\mathcal{F}(P)$ now is a set of subsets $\mathcal{N} \subset \mathcal{P}$. Thus, the map \mathcal{F} , together with the cost function \mathcal{G} can be interpreted as a directed weighted hypergraph (\mathcal{P}, E) with the set $E \subset \mathcal{P} \times 2^{\mathcal{P}}$ of hyperedges given by

$$E = \{(P, \mathcal{N}) : \mathcal{N} \in \mathcal{F}(P)\},$$

cf. Figure 3

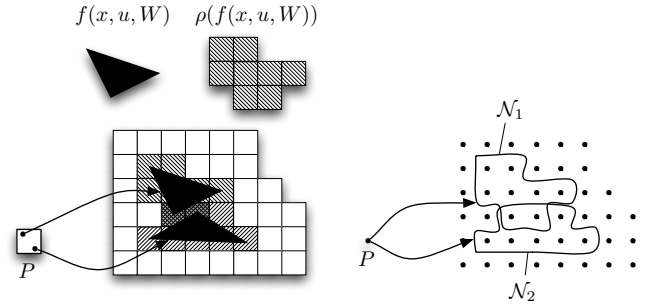


Bild 3: Partition of phase space, images of an element (left) and corresponding edges in the induced hypergraph (right).

4.3 A min-max shortest path algorithm

Unlike other computational methods for optimal control problems, it was shown in [9] that the main trick in Dijkstra's method, i.e., the reformulation of the minimization as a computationally efficiently solvable sorting problem, can be carried over to the min-max setting without increasing the computational complexity, see [14] for details.

For $\mathcal{T} := \{P \in \mathcal{P} : P \cap T \neq \emptyset\}$, the algorithm reads:

Algorithm 1. MIN-MAX DIJKSTRA($(\mathcal{P}, E), \mathcal{G}, \mathcal{T}$)

- 1 for each $P \in \mathcal{P}$ set $V(P) := \infty$
- 2 for each $P \in \mathcal{T}$ set $V(P) := 0$
- 3 $\mathcal{Q} := \mathcal{P}$
- 4 while $\mathcal{Q} \neq \emptyset$
- 5 $P := \operatorname{argmin}_{P' \in \mathcal{Q}} V(P')$
- 6 $\mathcal{Q} := \mathcal{Q} \setminus \{P\}$
- 7 for each $(Q, \mathcal{N}) \in E$ with $P \in \mathcal{N}$
- 8 if $\mathcal{N} \subset \mathcal{P} \setminus \mathcal{Q}$ then
- 9 if $V(Q) > \mathcal{G}(Q, \mathcal{N}) + V(P)$ then
- 10 $V(Q) := \mathcal{G}(Q, \mathcal{N}) + V(P)$

4.4 Convergence

It is natural to ask whether the approximate value function converges to the true one when the element diameter of the underlying partition goes to zero. This has be

proven pointwise on the stabilizable set S in the unperturbed case [12], as well as in an L^1 -sense on S and an L^∞ sense on the domain of continuity in the perturbed case, assuming continuity of V on the boundary of the target set T [9].

5 The discretization as a perturbation

It was shown in [6] that the feedback (6) using the approximate value function $V_{\mathcal{P}}$ from (8) will practically stabilize the system under suitable conditions. Numerical experiments (cf. Section 5.5), however, reveal that typically a rather fine partition is needed in order to achieve stability of the closed loop system. Furthermore, even on this fine partition, the approximate value function does not decay monotonically along system trajectories. The reason is that the approximate value function is rather heavily underestimated by the discretization scheme from Section 3.2. Whenever the trajectory of the closed loop system enters a certain element, it may be impossible to reach another element with a lower value from the current state (but it is possible when starting from another state in the same element). Formally, this is reflected by the two inequalities

$$V_{\mathcal{P}}(x) \leq V(x) \quad (18)$$

and

$$\begin{aligned} V_{\mathcal{P}}(x) &\leq \min_{u \in U} \{g(x, u) + V_{\mathcal{P}}(f(x, u))\} \\ &= g(x, u_{\mathcal{P}}(x)) + V_{\mathcal{P}}(f(x, u_{\mathcal{P}}(x))). \end{aligned} \quad (19)$$

A comparison with the Lyapunov function property (7) reveals that inequality (19) delivers exactly the *opposite* than what is needed in order to prove asymptotic stability.

5.1 Construction of the dynamic game

In order to cope with this phenomenon we are going to use the dynamic game formulation outlined above. The idea is to additionally incorporate the discretization error as a perturbation of the original control system. More precisely, in our context, instead of dealing with the single-valued system $x \mapsto f(x, u(x))$, we consider the multi-valued system $x \mapsto f(\rho(x), u(x))$. When computing the value of a given state x under the multi-valued dynamics, one assumes the “worst case” and sums the one step cost $g(x, u(x))$ with the *maximum* of V over the set $f(\rho(x), u(x))$.

Let us be more precise: given a control system (3) and an associated cost function g , we consider the dynamic game

$$F(x, u, W) := f(\rho(x), u) \quad (20)$$

for $x \in X, u \in U$ and define

$$G(x, u) = \sup_{x' \in \rho(x)} g(x', u). \quad (21)$$

5.2 Properties of the approximate value function

The following theorem shows the crucial properties of the associated approximate value function $V_{\mathcal{P}}$, in particular, that this function satisfies the opposite inequalities compared to (18), (19), when the terminal set T is appropriately included in the formulation.

Theorem 1 ([8]). *Let V denote the optimal value function of the optimal control problem described in Section 3.1 and let $V_{\mathcal{P}}$ denote the approximate optimal value function of the game F, G on a given partition \mathcal{P} with target set $T \subset \mathcal{P}$ with $0 \in T$. Then,*

$$V(x) - \max_{y \in T} V(y) \leq V_{\mathcal{P}}(x), \quad (22)$$

i.e. $V_{\mathcal{P}}$ is an upper bound for $V - \max V|_T$. Furthermore, $V_{\mathcal{P}}$ satisfies

$$V_{\mathcal{P}}(x) \geq \min_{u \in U} \{g(x, u) + V_{\mathcal{P}}(f(x, u))\} \quad (23)$$

for all $x \in X \setminus T$.

5.3 The feedback is the shortest path

Define

$$u_{\mathcal{P}}(x) = \operatorname{argmin}_{u \in U} \left\{ G(x, u) + \sup_{x' \in f(\rho(x), u)} V_{\mathcal{P}}(x') \right\}.$$

Due to the construction of the game F, G , this feedback is actually constant on each partition element. Moreover, we can directly extract $u_{\mathcal{P}}$ from the min–max Dijkstra Algorithm 1, for details we refer to [8].

Once $u_{\mathcal{P}}$ has been computed for every partition element, the only online computation that remains to do be done is the determination of the partition element for each state on the feedback trajectory. As described in [9] this can be done efficiently by storing the partition in a binary tree leading to a number of online computations which is logarithmic in the number of partition elements.

5.4 Behavior of the closed loop system

Finally, we can now state the properties of the feedback law $u_{\mathcal{P}}$ constructed in the last section.

Theorem 2 ([8]). *Under the assumptions of Theorem 1, if $(x_k)_k$ denotes the trajectory of the closed loop system with feedback $u_{\mathcal{P}}$ and if $V_{\mathcal{P}}(x_0) < \infty$, then there exists $k^* \in \mathbb{N}$ such that for $k = 0, \dots, k^* - 1$*

$$V_{\mathcal{P}}(x_k) \geq g(x_k, u_{\mathcal{P}}(x_k)) + V_{\mathcal{P}}(x_{k+1}),$$

and $x_{k^} \in T$.*

If the system (3) is asymptotically controllable to the origin and V is continuous, then we can use the same arguments as in [9] in order to show that on increasingly finer partitions \mathcal{P}_ℓ and for targets T_ℓ shrinking down to $\{0\}$ we obtain $V_{\mathcal{P}_\ell} \rightarrow V$.

5.5 Example: an inverted pendulum

The model. In order to illustrate the benefit of the approach proposed in this section, we consider the classical inverted pendulum on a cart, cf. [12, 6]. Let $\varphi \in [0, 2\pi]$ denote the angle between the pendulum and the upright vertical. Ignoring the dynamics of the cart, a model of the system is given by

$$\left(\frac{4}{3} - m_r \cos^2 \varphi\right) \ddot{\varphi} + \frac{m_r}{2} \dot{\varphi}^2 \sin 2\varphi - \frac{g}{\ell} \sin \varphi = -u \frac{m_r}{m\ell} \cos \varphi,$$

where we have used the parameters $m = 2$ for the pendulum mass, $m_r = m/(m + M)$ for the mass ratio with cart mass $M = 8$, $\ell = 0.5$ as the length of the pendulum and $g = 9.8$ for the gravitational constant. As the cost function, we choose

$$q(\varphi, \dot{\varphi}, u) = \frac{1}{2} (0.1\varphi^2 + 0.05\dot{\varphi}^2 + 0.01u^2). \quad (24)$$

Denoting the system's evolution operator for constant control functions u by $\Phi^t(x, u)$, we consider the discrete time system (3) with $f(x, u) = \Phi^T(x, u)$ for $T = 0.1$, i.e., the sampled continuous time system with sampling period $T = 0.1$. The discrete time cost function is obtained by numerically integrating the continuous time instantaneous cost according to $g(\varphi, \dot{\varphi}, u) = \int_0^T q(\Phi^t((\varphi, \dot{\varphi}), u), u) dt$.

Implementation. We use the classical Runge-Kutta scheme of order 4 with step size 0.02 in order to approximate Φ^T . We choose $X = [-8, 8] \times [-10, 10]$ as the region of interest, $U = [-64, 64]$ and employ a box partition consisting of identically sized boxes for the discretization. In constructing the (hyper)graph (10) resp. (17) we approximate the graph map \mathcal{F} and the edge weights by integrating 9 equally spaced points on each partition box, choosing from 16 equally spaced values in U and minimizing over the associated values of g .

Results. Figure 4 shows the approximate optimal value function as well as the feedback trajectory for the initial value (3.1, 0.1) computed by the approach from Section 3.2 on a partition of 2^{18} boxes. As Figure 5 clearly shows, the approximate optimal value function does not decrease monotonically along this trajectory. This is due to the fact that this $V_{\mathcal{P}}$ does not satisfy the Lyapunov inequality (7). In particular, on a coarser partition of 2^{14} boxes, the associated feedback is not stabilizing this initial condition any more.

In contrast to this, the approach from this section yields an approximation $V_{\mathcal{P}}$ which does satisfy the Lyapunov inequality (7) outside T and hence an approximation to V which is a Lyapunov function itself. Figure 6 shows the approximate upper value function on a partition of 2^{14} boxes with target set $T = [-0.1, 0.1]^2$ as well as the trajectory generated by the associated feedback for the

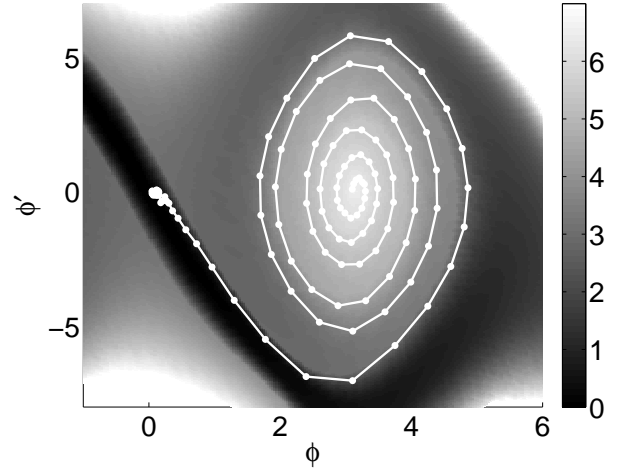


Bild 4: Approximate optimal value function and feedback trajectory for the inverted pendulum using the approach from Section 3.2.

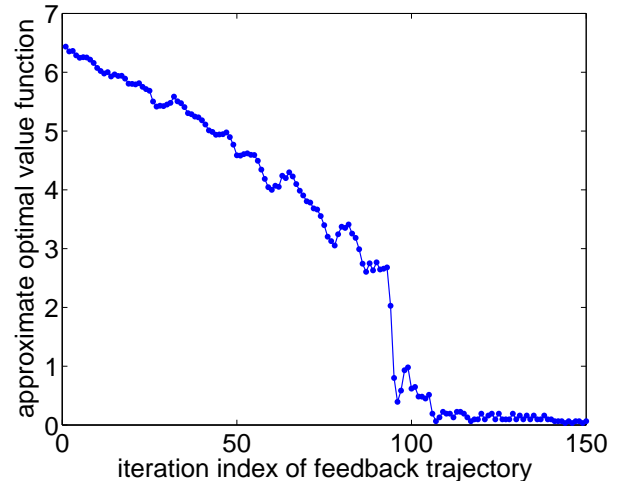


Bild 5: Approximate optimal value function along the feedback trajectory for the inverted pendulum using the approach from Section rebsubsec:naive.

initial value (3.1, 0.1). As expected, the approximate value function is decreasing monotonically along this trajectory. Furthermore, despite the fact that we used considerably fewer boxes as for Figure 4, the resulting trajectory is obviously closer to the optimal one because it converges to the origin much faster.

It should be remarked that in order to construct the graph map \mathcal{F} a numerical evaluation of the set valued images $f(P, u)$ needs to be done, which is a serious problem at its own right. Typically (as mentioned above) the set $f(P, u)$ is being approximated by mapping a certain number of sample points in P . In the case that one is interested in a *rigorous* computation, either techniques based on Lipschitz estimates for f [11] or interval arithmetic [3] can be employed.

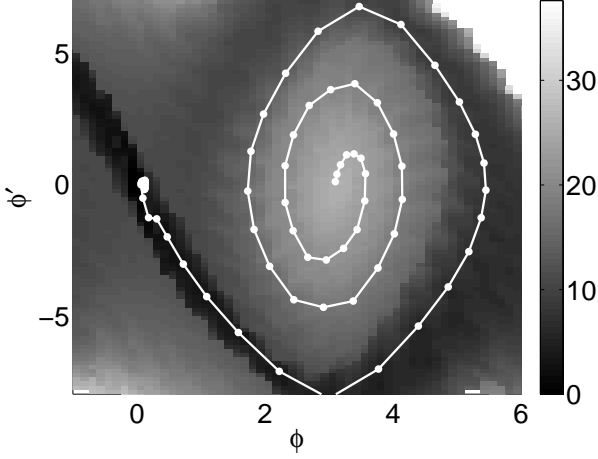


Bild 6: Approximate upper value function and feedback trajectory for the inverted pendulum using the dynamic game feedback construction.

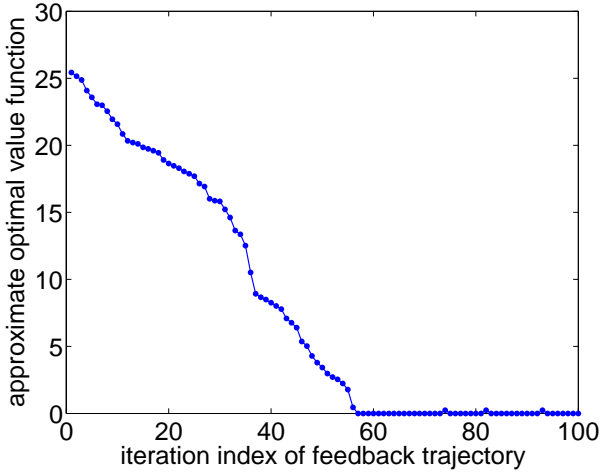


Bild 7: Approximate upper value function along the feedback trajectory for the inverted pendulum using the dynamic game feedback construction.

6 Hybrid systems

The discretization approach described in Sections 3.2 and 4.2 is naturally suited for the treatment of *hybrid systems*. In order to exemplify this, we consider the problem of optimally stabilizing the continuous state component x of a discrete-time nonlinear hybrid control system given by

$$\begin{aligned} x_{k+1} &= f_c(x_k, y_k, u_k) \\ y_{k+1} &= f_d(x_k, y_k, u_k) \end{aligned} \quad k = 0, 1, \dots, \quad (25)$$

with continuous state dynamics $f_c : X \times Y \times U \rightarrow X \subset \mathbb{R}^n$ and discrete state dynamics $f_d : X \times Y \times U \rightarrow Y$. Here the set U of possible control inputs is finite³, the set $X \subset \mathbb{R}^n$ of continuous states is compact and the set Y of discrete states (or modes) is an arbitrary finite set. The

³ If desired, continuous control values could also be included and treated with the discretization technique described in [12, 6].

solutions of (25) for initial values $x_0 = x$, $y_0 = y$ and control sequence $\mathbf{u} = (u_0, u_1, \dots) \in U^{\mathbb{N}}$ are denoted by $x_k(x, y, \mathbf{u})$ and $y_k(x, y, \mathbf{u})$, respectively, and we assume that for each $k \geq 0$ the map $x_k(\cdot, y, \mathbf{u})$ is continuous for each $y \in Y$ and each $\mathbf{u} \in U^{\mathbb{N}}$. Note that if f_d does not depend on x , then this is equivalent to $f_c(\cdot, y, u) : X \times Y \rightarrow \mathbb{R}^d$ being continuous for each $y \in Y$, $u \in U$.

Problem formulation. Similar to the previous sections, given a target set $T \subset X$, the goal of the optimization problem we want to solve is to find a control sequence $u_k, k = 0, 1, 2, \dots$, such that $x_k \rightarrow T$ as $k \rightarrow \infty$, while minimizing the accumulated cost $g : X \times Y \times U \rightarrow [0, \infty)$ with $g(x, y, u) > 0$ for all $x \notin T$, $y \in Y$ and $u \in U$.

To this end, we would like to construct an approximate *optimal feedback* $u : S \rightarrow U$ such a suitable approximate asymptotic stability property for the resulting closed loop system holds. Again, the construction will be based on an approximation of the (optimal) value function which will act as a Lyapunov function. For an appropriate choice of g this function is continuous in x at least in a neighborhood of T [10].

6.1 Computational approach

Let \mathcal{Q} be a *partition* of the continuous state set X , that is a finite collection of compact subsets $Q_i \subset X$, $i = 1, \dots, r$, with $\cup_{i=1}^r Q_i = X$, and $m(Q_i \cap Q_j) = 0$ for $i \neq j$ (where m denotes Lebesgue measure). Then the sets

$$\mathcal{P} := \{Q_i \times \{y\} \mid Q_i \in \mathcal{Q}, y \in Y\} \quad (26)$$

form a partition of the product state space $Z = X \times Y$. On \mathcal{P} the approaches from Sections 3.2 and 4.2 can be applied literally.

6.2 Example: A switched voltage controller

We reconsider an example from [13]: A switched power controller for DC to DC conversion. Within the controller, a semiconductor device is switching the polarity of a voltage source in order to keep the load voltage as constant as possible. The model is

$$\begin{aligned} \dot{x}_1 &= \frac{1}{C} (x_2 - I_{load}) \\ \dot{x}_2 &= -\frac{1}{L} x_1 - \frac{R}{L} x_2 + \frac{1}{L} u V_{in} \\ \dot{x}_3 &= V_{ref} - x_1, \end{aligned} \quad (27)$$

where $u \in \{-1, 1\}$ is the control input. In the following numerical experiment we use the same parameter values as given in [13].

The corresponding discrete time system is given by the time- h -map ϕ^h ($h = 0.1$ in our case) of (27), with the control input held constant during this sample period. The cost function is

$$g(x, u) = q_P(\phi_1^h(x) - V_{ref}) + q_D(\phi_2^h(x) - I_{load}) + q_I \phi_3^h(x).$$

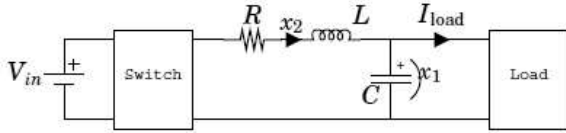


Bild 8: A switched DC/DC converter (taken from [13]).

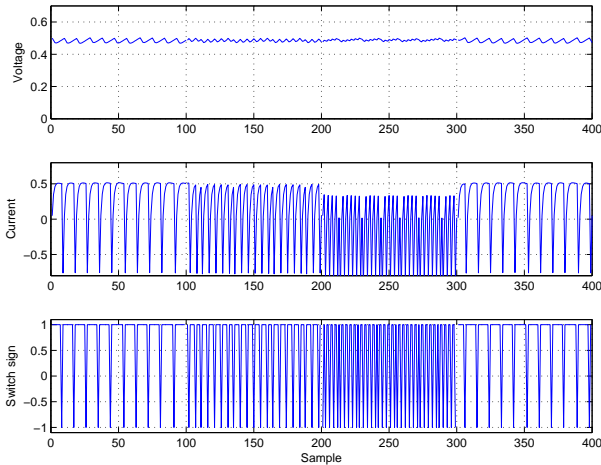


Bild 9: Simulation of the controlled switched power converter.

The third component in (27) is only being used in order to penalize a large L^1 -error of the output voltage. We slightly simplify the problem (over its original formulation in [13]) by using $x_3 = 0$ as initial value in each evaluation of the discrete map. Correspondingly, the map reduces to a two-dimensional one.

Confining our domain of interest to the rectangle $X = [0, 1] \times [-1, 1]$, our target set is given by $T = \{V_{ref}\} \times [-1, 1]$. For the construction of the finite graph, we employ a partition of X into 64×64 equally sized boxes. We use 4 test points in each box, namely their vertices, in order to construct the edges of the graph.

Using the resulting approximate optimal value function (associated to a nominal $I_{load} = 0.3$ A) and the associated feedback, we repeated the stabilization experiment from [13], where the load current is changed after every 100 iterations. Figure 6.2 shows the result of this simulation, proving that our controller stabilizes the system as requested.

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