Rigorous discretization of subdivision techniques

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Abstract

Recently efficient set-oriented methods have been proposed for the numerical investigation of dynamical systems [1, 2, 3]. A basic question arising in implementing these methods is to compute the "set-wise image" of some set: determine all sets of some collection which intersect the image of the given set. We describe how this discretization question can be tackled rigorously and furthermore how the numerical effort can be reduced to a minimum.

In [1, 2] set-oriented methods for the approximation of invariant sets of a discrete dynamical system $f : \mathbb{R}^n \to \mathbb{R}^n$ have been proposed. The idea is to use a hierarchy of box-collections in order to cover the invariant set of interest. Given some finite collection $\mathcal{B}$ of *boxes* $B = B(c,r) = \{x : |x-c| \le r\}$ (where we write $|x| = (|x_1|, \dots, |x_n|)$ and $x \le y$ for $x, y \in \mathbb{R}^n$, if $x_i \le y_i$ for $i = 1, \dots, n$), a crucial computational step within these algorithms is to compute the *set-wise image*

$$\mathcal{F}(B) = \{B' \in \mathcal{B} \mid f(B) \cap B' \neq \emptyset\},$$

for every $B \in \mathcal{B}$. One way to approximate $\mathcal{F}(B)$ for a given box $B = B(c,r) \in \mathcal{B}$ is to choose a (finite) set $T \subset B$ of *testpoints* and hope that

$$\mathcal{F}_T(B) = \{B' \in \mathcal{B} \mid f(T) \cap B' \neq \emptyset\}$$

provides a good approximation to $\mathcal{F}(B)$. In any case one has $\mathcal{F}_T(B) \subset \mathcal{F}(B)$, but not for every choice of T also the inverse inclusion holds. In numerical experiments the following heuristic choice has been shown to work well: we choose a small number of testpoints on every edge of B plus the center of the box (see [2] for more details on other heuristic choices of T).

Our aim here is to construct a set $\hat{\mathcal{F}}(B)$ of boxes for which

$$\mathcal{F}(B) \subset \hat{\mathcal{F}}(B),$$

so that we get a rigorous covering of $f(B)$. To this end we will need to know *local Lipschitz constants* for f, that is we require that for every box B in the current collection there is a nonnegative matrix $L = L(B) \in \mathbb{R}^{n \times n}$ such that

$$|f(y) - f(x)| \le L|y - x| \tag{1}$$

for $x, y \in B$. If f is continuously differentiable then $L_{ij} = \max_{\xi \in B} |\partial_j f_i(\xi)|$. Now let $h = h(B) \in \mathbb{R}^n$ be a positive vector such that

$$Lh \le 2r$$

Using the *meshwidths* h we now define a mesh

$$\hat{T} = \hat{T}(B) = \{x : (x_i - c_i) \in h_i\mathbb{Z}, i = 1, \dots, n\}.$$

It is easy to see that for every $y \in B$ there is a meshpoint $x \in \hat{T}(B)$, such that $|y - x| \le h/2$. On the other hand we are interested in a *finite* set of testpoints and indeed the only points $x \in \hat{T}(B)$ we really need are those for which there is actually a $y \in B$ with $|y - x| \le h/2$. So let

$$T(B) = \hat{T}(B) \cap \{x \mid B \cap \text{int } B(x, h/2) \neq \emptyset\}$$

be the set of testpoints. Note that an additional constraint on h will be necessary in order to ensure that the testpoints are contained in B, which is necessary, since the local Lipschitz-estimate (1) on f is only valid for points in B. Finally we construct the collection $\hat{\mathcal{F}}(B)$ by setting

$$\hat{\mathcal{F}}(B) = \{\hat{B} \in \mathcal{B} \mid \hat{B} \cap B(f(x), r) \neq \emptyset \text{ for some } x \in T(B)\}. \tag{2}$$

The idea of this construction is to look at the boxes $B(f(x), r) \notin \mathcal{B}$ corresponding to the images of the testpoints x and to collect in $\hat{\mathcal{F}}(B)$ all boxes which have nonempty intersection with those boxes, cp. Figure 1. It is important to note that the construction of $\hat{\mathcal{F}}(B)$ is a finite task: since the boxes $B(f(x), r)$, $x \in T$, have the same radius as the boxes in the collections $\mathcal{B}$, it suffices to consider the vertices of $B(f(x), r)$.

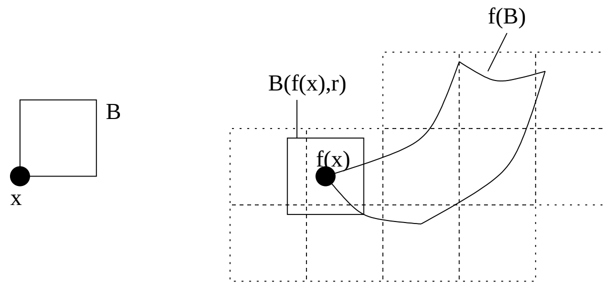

Figure 1: On the construction of $\hat{\mathcal{F}}(B)$

Example: attractor of the Hénon-map

We will now compare the rigorous construction of $\hat{\mathcal{F}}(B)$ to the heuristic one of $\mathcal{F}_T(B)$ by computing a covering of the relative global attractor for the Hénon-map

$$f(x, y) = (1 - ax^2 + y/5, 5bx) \tag{3}$$

with parameters $b = 0.2$ and $a = 1.2$. We apply the subdivision algorithm [2] to the initial collection $\mathcal{B}_0 = \{[-2, 2]^2\}$.

The dark shaded boxes in Figure 2 represent the covering obtained by the heuristic choice of testpoints while the rigorous covering is given by all boxes shown. The following table compares the numerical effort (in terms of the number of function evaluations) of the rigorous versus the heuristic method. Evidently the rigorous construction is even more efficient than the heuristic one:

r	rigorous		heuristic, 13 points		heuristic, 121 points	
	f-eval.	$\lvert\mathcal{B}\rvert$	f-eval.	$\lvert\mathcal{B}\rvert$	f-eval.	$\lvert\mathcal{B}\rvert$
2^{-1}	886	43	1144	25	11132	26
2^{-2}	2214	96	2912	56	27830	58
2^{-3}	4960	216	6838	128	65824	138
2^{-4}	10476	476	14742	233	145926	258

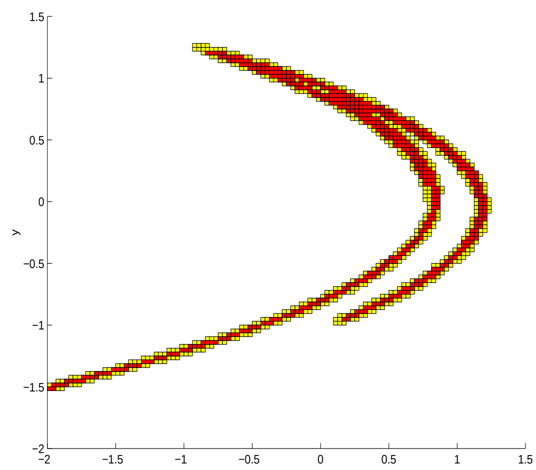

Figure 2: Rigorous covering of the relative global attractor for the Hénon-map

Adaptive choice of the testpoints

In order to reduce the numerical effort of the set-oriented algorithms one has to reduce the number of testpoints per box as far as possible. We now show how to do that by considering local expansion rates of the map f. To this end we consider the singular value decomposition

$$Df(x) = U(x)S(x)V^{\mathsf{T}}(x)$$

of $Df(x)$, $x \in B(c, r)$, where $U(x) = [u_1(x), \ldots, u_n(x)]$ and $V(x) = [v_1(x), \ldots, v_n(x)]$ are real orthogonal $(n \times n)$-matrices and $S(x) \in \mathbb{R}^{n \times n}$ is a diagonal matrix having the singular values

$\sigma_1(x) \geq \cdots \geq \sigma_n(x)$ of $Df(x)$ on the diagonal. The singular values are the lengths of the semiaxes of the ellipsoid $E = \{y' = Df(x)y \mid \|y\|_2 = 1\}$ and since

$$Df(x)v_i(x) = \sigma_i(x)u_i(x), \quad i = 1, \ldots, n,$$

$v_1(x)$ is the direction which is most expanded by $Df(x)$ (and turned to $u_1(x)$).

The idea for an improved choice of the testpoints is to construct a mesh with respect to the basis of right singular vectors $v_1(c), \ldots, v_n(c)$ of $Df(c) = U(c)S(c)V^\mathsf{T}(c)$ and to choose the meshwidth h_i, $i = 1, \ldots, n$ in relation to the singular value $\sigma_i(c)$. Let us suppose for the moment that for a box B the derivative $Df(x) = Df(c) = Df = USV^\mathsf{T}$ is constant on a sufficiently large neighborhood $\Delta(B)$ of B. Then

$$f(x) - f(y) = Df \cdot (x - y) = USV^\mathsf{T}(x - y),$$

so that

$$|f(x) - f(y)| \leq |U|S|V^T(x - y)|$$

where $|U| = (|u_{ij}|)$. We choose meshwidths $h \in \mathbb{R}^n$, $h > 0$, such that

$$|U|Sh \leq 2r \tag{4}$$

and define the mesh

$$\hat{T} = \hat{T}(B) = \{x : (V^T(x - c))_i \in h_i\mathbb{Z}, i = 1, \ldots, n\}. \tag{5}$$

Again it is easy to see that for every $y \in B$ there is a mesh point $x \in \hat{T}(B)$, such that $|V^T(y - x)| \leq h/2$. We have to restrict ourselves to a finite set of testpoints again which can be written down as

$$T = T(B) = \hat{T}(B) \cap \{x : B \cap \mathrm{int}\,(VB(0, h/2) + x) \neq \emptyset\}. \tag{6}$$

We construct $\hat{\mathcal{F}}(B)$ as in (2) and get that $\mathcal{F}(B) \subset \hat{\mathcal{F}}(B)$. Finally let us consider the general case where $Df(x)$ is not constant on a box. Let

$$M(x) = (m_{ij}(x))_{i,j=1,\ldots,n} = Df(x)V(c)$$

and set

$$\overline{M} = (\overline{m}_{ij})_{i,j=1,\ldots,n}, \quad \overline{m}_{ij} = \max_{x \in \Delta(B)} |m_{ij}(x)|,$$

where $\Delta(B)$ is a sufficiently large neighborhood of B which we suppose to be convex in this case. We choose meshwidths $h > 0$ such that

$$\overline{M}h \leq 2r, \tag{7}$$

and use the mesh as defined by (5) as well as the construction (2) for $\hat{\mathcal{F}}(B)$.

THEOREM 1 *The union of boxes in $\hat{\mathcal{F}}(B)$ covers $f(B)$, i.e.*

$$\mathcal{F}(B) \subset \hat{\mathcal{F}}(B).$$

References

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