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Delay-optimal global feedbacks for quantized networked event systems

Stefan Jerg* and Oliver Junge*
Center for Mathematical Sciences
Technische Universität München
85747 Garching, Germany
{jerg,junge}@ma.tum.de
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***Abstract*— We extend a recent optimization based technique for the construction of globally stabilizing optimal controllers for quantized nonlinear event systems in a digital network. To this end, we assume that the plant and the controller possess synchronized clocks and that at each event the (global) time stamp is transmitted from the event generator to the controller. The new construction explicitely incorporates the delay information, rendering the controller more robust. The method is illustrated by means of the inverted pendulum in a digital loop.**

I. INTRODUCTION

For controllers which are connected to the plant via a digital network it is often desirable to reduce the amount of information transmitted from the plant to the controller (and vice versa) as much as possible. To this end, *event based* control schemes have been devised, see, e.g. [2], [8], [1] which depart from the sampled data approach by transmitting state information only in dependence on whether an *event* occured. Events are often defined in terms of a *quantization* of state space such that an event is triggered whenever the state of the plant transits from one quantization cell into another.

Recently, optimization based schemes for the construction of globally stabilizing controllers for quantized event systems have been constructed [4] which in turn have been extended to the more natural situation that the plant-controller loop is subject to delayed or lost transmission of information in [7] (we refer to [5], [9] and [11] for the cases of linear and non-quantized systems, for non-linear systems with constant delay and for modeling losses without delays, respectively).

While in [7] delays have been modelled by random inputs and have been been taken into account via a stochastic optimality principle, here we propose to extend our approach from [7] by explicitely incorporating the delay information into the feedback construction. As expected, our numerical experiment for the inverted pendulum shows increased robustness in comparison to the stochastic feedback. In particular, the computed value function is a Lyapunov function for the closed loop system and the associated feedback stabilizes the system *independently of how the delays are chosen*.

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II. OPTIMAL FEEDBACKS FOR QUANTIZED EVENT SYSTEMS

We consider a plant which can be modelled by a nonlinear discrete time control system

$$\boldsymbol{x}(k+1) = f(\boldsymbol{x}(k), \boldsymbol{u}(k)), \quad k = 0, 1, 2, \dots, \tag{1}$$

where $f : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ is continuous, $\boldsymbol{x}(k) \in \mathcal{X}$ is the state and $u(k) \in \mathcal{U}$ is the control input. In addition to f, we are given a continuous running cost function $c : \mathcal{X}\times\mathcal{U} \to [0, \infty)$ as well as a *target set* $\mathcal{X}^* \subset \mathcal{X}$. Our goal is to compute a feedback law $\boldsymbol{u} : \mathcal{X} \to \mathcal{U}$ for this system which drives the system towards the target set $\mathcal{X}^*$ while accumulating the least costs possible.

The information which is transmitted from the plant to the controller is restricted in the following two ways:

1) The controller only receives information on the state whenever an *event* occurs. Formally we are dealing with the discrete time system

$$\boldsymbol{x}(\ell+1) = \tilde{f}(\boldsymbol{x}(\ell), \boldsymbol{u}(\ell)), \quad \ell = 0, 1, \dots,$$

where $\tilde{f}(\boldsymbol{x}, \boldsymbol{u}) = f^{r(\boldsymbol{x},\boldsymbol{u})}(\boldsymbol{x}, \boldsymbol{u})$, the function $r : \mathcal{X} \times \mathcal{U} \to \mathbb{N}_0$ is a given *event function* and the iterate f^r is defined by $f^0(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{x}$ and $f^r(\boldsymbol{x}, \boldsymbol{u}) = f(f^{r-1}(\boldsymbol{x}, \boldsymbol{u}), \boldsymbol{u})$, cf. [4]. Accordingly, we define an associated running cost $\tilde{c} : \mathcal{X} \times \mathcal{U} \to [0, \infty)$ by $\tilde{c}(\boldsymbol{x}, \boldsymbol{u}) = \sum_{k=1}^{r} c(f^k(\boldsymbol{x}, \boldsymbol{u}), \boldsymbol{u})$. Note that we can reconstruct the "true time" k from the "event time" ℓ by the event function r: we have that $k(\ell + 1) = k(\ell) + r(\boldsymbol{x}(\ell), \boldsymbol{u}(\ell))$.

2) The controller only receives *quantized* information on the state. Formally, we are given a (finite) partition $P = \{\mathcal{P}_1, \dots, \mathcal{P}_d\}$, $\mathcal{P}_i \subset \mathcal{X}$, of $\mathcal{X}$ which induces a equivalence relation $\sim$ on $\mathcal{X}\times\mathcal{X}$ by $x \sim y \Leftrightarrow x$ and y lie in the same partition element. We denote by $[\boldsymbol{x}] \in P$ the corresponding equivalence class of $\boldsymbol{x} \in \mathcal{X}$. Only $[\boldsymbol{x}(k)]$ is transmitted from the plant to the controller. Thus, from the viewpoint of the controller, the plant is given by the finite state system, cf. [3], [4]

$$\mathcal{P}(\ell+1) = F(\mathcal{P}(\ell), \boldsymbol{u}(\ell), \gamma(\ell)), \quad \ell = 0, 1, \dots, \tag{2}$$

defined by $F(\mathcal{P}, \boldsymbol{u}, \gamma) = [\tilde{f}(\gamma(\mathcal{P}, \boldsymbol{u}), \boldsymbol{u})]$, $\mathcal{P} \in P, \boldsymbol{u} \in \mathcal{U}$, where $\gamma : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ denotes a *choice function* which satisfies $[\gamma(\mathcal{P}, \boldsymbol{u})] = \mathcal{P}$ for all $\mathcal{P} \in P$ and all

$\boldsymbol{u} \in \mathcal{U}$. The choice function models the fact that it is unknown to the controller from which exact state $\boldsymbol{x}(\ell)$ the system transits to the next cell $\mathcal{P}(\ell+1)$. We let $\mathcal{C}$ denote the set of those functions. Formally, (2) constitutes a *dynamic game*.

III. Computing the optimal feedback.

In order to be compatible with our quantization, from now on we assume that $\mathcal{X}^*$ is a union of cells from X. For the quantized system (2) we define

$$C(\mathcal{P},\boldsymbol{u}) = \sup_{\boldsymbol{x}\in\mathcal{P}} \tilde{c}(\boldsymbol{x},\boldsymbol{u}).$$

For $\mathcal{P}_0 \in P$, $\underline{\boldsymbol{u}} = (\boldsymbol{u}(\ell))_\ell \in \mathcal{U}^{\mathbb{N}}$ and $\underline{\gamma} = (\gamma(\ell))_\ell \in \mathcal{C}^{\mathbb{N}}$, the cost accumulated along a trajectory $(\mathcal{P}(\ell))_\ell \in P^{\mathbb{N}}$ of (2) is

$$J(\mathcal{P}(0),\underline{\boldsymbol{u}},\underline{\gamma}) = \sum_{\ell=0}^{N} C(\mathcal{P}(\ell),\boldsymbol{u}(\ell)),$$

where $N = \min\{\ell \geq 0 : \mathcal{P}(\ell) \subset \mathcal{X}^*\}$. The *optimal value function* is

$$V(\mathcal{P}) = \sup_{\underline{\gamma}\in\mathcal{C}^{\mathbb{N}}} \inf_{\underline{\boldsymbol{u}}\in\mathcal{U}^{\mathbb{N}}} J(\mathcal{P},\underline{\boldsymbol{u}},\underline{\gamma}),$$

which – by standard arguments – is the unique solution to the *optimality principle*

$$V(\mathcal{P}) = \inf_{\boldsymbol{u}\in\mathcal{U}} \{C(\mathcal{P},\boldsymbol{u}) + \sup_{\gamma\in\mathcal{C}} V(F(\mathcal{P},\boldsymbol{u},\gamma))\}$$

together with the boundary condition $V|_{\mathcal{X}^*} = 0$.

Given V, we compute an optimal feedback for (2) resp. (1) by setting

$$u(\boldsymbol{x}) = \operatorname*{argmin}_{\boldsymbol{u}\in\mathcal{U}} \left\{ C([\boldsymbol{x}],\boldsymbol{u}) + \sup_{\gamma\in\mathcal{C}} V(F([\boldsymbol{x}],\boldsymbol{u},\gamma)) \right\}.$$

For a given initial state $\boldsymbol{x}$ and a sequence $\underline{\gamma}$ of choice functions, this feedback generates a control sequence $\underline{\boldsymbol{u}}$ which minimizes J. The associated trajectory $(\mathcal{P}(\ell))_\ell$ of (2) is the *shortest path* from $[\boldsymbol{x}]$ to $\mathcal{X}^*$ in the hypergraph $G = (P,\mathcal{E})$, where the edges of G are given by

$$\mathcal{E} = \{(\mathcal{P},N) : \mathcal{P}\in P, N = F(\mathcal{P},\boldsymbol{u},\mathcal{C}), \boldsymbol{u}\in\mathcal{U}\}$$

weighted by

$$w(\mathcal{P},N) = \inf\{C(\mathcal{P},\boldsymbol{u}) : \boldsymbol{u}\in\mathcal{U}, F(\mathcal{P},\boldsymbol{u},\mathcal{C}) = N\}.$$

As such, it can be computed by an efficient Dijkstra-type algorithm, cf. [3].

IV. Extension to networked systems with delays and losses.

In the case that the transmission of the events from the plant to the controller is subject to delays and losses (i.e. events arrived delayed or not at all at the controller), the plant will still operate for some time with the old control input computed from the previous event. Depending on the length of the delay and the old input, the control input for the next event will then have to be suitably adapted. In order to model this situation, in [7] a stochastic model has been proposed which accounts for the effects of the delay using a stochastic optimality principle.

Here, our goal is to derive an optimal controller which makes use of fact that at each event the delay of the transmitted information is known to the controller (since the transmissions are time-stamped and plant and controller possess synchronized clocks). To this extent, we augment the state of the system by the current delay, $\boldsymbol{z} = (\boldsymbol{x},\boldsymbol{w},\delta) \in \mathcal{Z} := \mathcal{X}\times\mathcal{U}\times\mathcal{D}$, with $\boldsymbol{x}\in\mathcal{X}$ the current state, $\boldsymbol{w}\in\mathcal{U}$ the old control input and $\delta\in\mathcal{D} = \{1,2,\ldots,\delta_{\text{max}}\}\cup\{\infty\}$ the current delay and consider the control system

$$\boldsymbol{z}(\ell+1) = g(\boldsymbol{z}(\ell),\boldsymbol{u}(\ell),\sigma(\ell)), \quad \ell = 0,1,2,\ldots, \tag{3}$$

with

$$g(\boldsymbol{z},\boldsymbol{u},\sigma) = \begin{bmatrix} f^s\left(f^t(\boldsymbol{x},\boldsymbol{w}),\boldsymbol{u}\right) \\ \boldsymbol{w}' \\ \sigma(\boldsymbol{z}) \end{bmatrix},$$

where

$$t = t(\boldsymbol{z}) = \min\{\delta, r(\boldsymbol{x},\boldsymbol{w})\},$$
$$s = s(\boldsymbol{z}) = \begin{cases} r\left(f^t(\boldsymbol{x},\boldsymbol{w}),\boldsymbol{u}\right) & \text{if } \delta < r(\boldsymbol{x},\boldsymbol{w}), \\ 0 & \text{otherwise,} \end{cases}$$
$$\boldsymbol{w}' = \begin{cases} \boldsymbol{u} & \text{if } \delta < r(\boldsymbol{x},\boldsymbol{w}), \\ \boldsymbol{w} & \text{otherwise,} \end{cases}$$

and $\sigma : \mathcal{Z}\to\mathcal{D}$ is a function which encodes the information about the delay of the next event. In this model, any delay $\delta \geq r(\boldsymbol{x},\boldsymbol{w})$ is treated as $\delta = \infty$, i.e. as if the corresponding data would never reach the controller.

A. Quantization model

Since the controller only receives quantized information on the state, from the viewpoint of the controller the plant is given by a finite state dynamic game

$$\mathcal{Q}(\ell+1) = G(\mathcal{Q}(\ell),\boldsymbol{u}(\ell),\gamma(\ell)), \quad \ell = 0,1,\ldots, \tag{4}$$

defined by $G(\mathcal{Q},\boldsymbol{u},\gamma) = [g(\eta(\mathcal{Q},\boldsymbol{u}),\boldsymbol{u},\sigma)]$, where $\mathcal{Q} = (\mathcal{P},\boldsymbol{w},\delta) \in Z := P\times\mathcal{U}\times\mathcal{D}$ is a partition element in the extended state space, $\gamma = (\eta,\sigma)$, $\sigma : \mathcal{Z}\to\mathcal{D}$ and $\eta : Z\times\mathcal{U}\to Z$ is a choice function satisfying $[\eta(\mathcal{Q},\boldsymbol{u})] = \mathcal{Q}$. Let Γ denote the space of those functions γ.

B. Optimality principle and optimal feedback

For the quantized delay system (4), the costs accumulated along a trajectory are

$$J(\mathcal{Q}(0),\underline{\boldsymbol{u}},\underline{\gamma}) = \sum_{\ell=0}^{N} C(\mathcal{P}(\ell),\boldsymbol{u}(\ell)),$$

with $\mathcal{Q} = (\mathcal{P},\boldsymbol{w},\delta)$ and $N = \min\{\ell \geq 0 : \mathcal{Q}(\ell) \subset \mathcal{X}^*\times\mathcal{U}\times\mathcal{D}\}$. The corresponding *upper optimal value function* is

$$V(\mathcal{Q}) = \sup_{\underline{\gamma}\in\Gamma^{\mathbb{N}}} \inf_{\underline{\boldsymbol{u}}\in\mathcal{U}^{\mathbb{N}}} J(\mathcal{Q},\underline{\boldsymbol{u}},\underline{\gamma}).$$

Again, by standard arguments, V fulfills the optimality principle

$$V(\mathcal{Q}) = \inf_{\boldsymbol{u}\in\mathcal{U}} \left\{ C(\mathcal{P},\boldsymbol{u}) + \sup_{\gamma\in\Gamma} V(G(\mathcal{Q},\boldsymbol{u},\gamma)) \right\}$$

($\mathcal{Q} = (\mathcal{P}, \boldsymbol{w}, \delta)$), together with the boundary condition $V|_{\mathcal{X}^* \times \mathcal{U} \times \mathcal{D}} = 0$ and we obtain an optimal feedback for (3) by

$$u(\boldsymbol{z}) = \underset{\boldsymbol{u} \in \mathcal{U}}{\operatorname{argmin}} \{C([\boldsymbol{x}], \boldsymbol{u}) + \sup_{\gamma \in \Gamma} V(G([\boldsymbol{z}], \boldsymbol{u}, \gamma)\},$$

where $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{w}, \delta)$. Note that in contrast to the construction in [7], the information on the current delay is explicitely taken into account here.

V. Numerical experiment

For our numerical experiments we consider a version of the classical inverted pendulum on a cart, cf. [6]. The nonlinear system is given by the continuous time control system

$$\left(\tfrac{4}{3} - m_r \cos^2 \varphi\right) \ddot{\varphi} + \tfrac{m_r}{2} \dot{\varphi}^2 \sin 2\varphi - \tfrac{g}{\ell} \sin \varphi = -u \, \tfrac{m_r}{m\ell} \cos \varphi,$$

where we have used the parameters $m = 2$ for the pendulum mass, $m_r = m/(m+M)$ for the mass ratio with cart mass $M = 8$, $\ell = 0.5$ as the length of the pendulum and $g = 9.8$ for the gravitational constant. The instantaneous cost is

$$q(\varphi, \dot{\varphi}, u) = \frac{1}{2}\left(0.1\varphi^2 + 0.05\dot{\varphi}^2 + 0.01u^2\right). \qquad (5)$$

Denoting the evolution operator of the system for constant control functions $u(t) \equiv \boldsymbol{u}$ by $\boldsymbol{\Phi}^t(\boldsymbol{x}, \boldsymbol{u})$, $\boldsymbol{x} = (\varphi, \dot{\varphi})$, we consider the discrete time system $f(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{\Phi}^T(\boldsymbol{x}, \boldsymbol{u})$ for $T = 0.01$, i.e., the sampled continuous time system with sampling rate $T = 0.01$. The map $\boldsymbol{\Phi}^T$ is approximated via the classical Euler scheme with step size 0.01. The discrete time cost function is obtained by numerically integrating the continuous time instantaneous cost according to $c(\boldsymbol{x}, \boldsymbol{u}) = \int_0^T q(\boldsymbol{\Phi}^t(\boldsymbol{x}, \boldsymbol{u}), \boldsymbol{u}) dt$. We choose $\mathcal{X} = S^1 \times [-8, 8]$ (i.e. angle modulo complete rotations of the pendulum) as the state space and $\mathcal{X}^* = [-\frac{\pi}{8}, \frac{\pi}{8}] \times [-\frac{3}{4}, \frac{3}{4}]$ as the target region (8×6 boxes of a $2^6 \times 2^6$ box grid of $\mathcal{X}$). By means of this $2^6 \times 2^6$ grid of $\mathcal{X}$ we define the event function as follows. Let $\mathbf{s}(\boldsymbol{x})$ and $\mathbf{t}(\boldsymbol{x})$ denote the center and the radius of the rectangular box containing $\boldsymbol{x}$, respectively. Then by means of the *event set*

$$\beta(\boldsymbol{x}) = \{y \in \mathcal{X} : |y_i - s_i(\boldsymbol{x})| \le e_r t_i(\boldsymbol{x}), i = 1, 2\} \qquad (6)$$

with event radius $e_r = 9.4$ we define the event function

$$r(\boldsymbol{x}, \boldsymbol{u}) = \min\{t \in 0.01\mathbb{N} : \boldsymbol{\Phi}^t(\boldsymbol{x}, \boldsymbol{u}) \notin \beta(\boldsymbol{x})\} \qquad (7)$$

with $\min \emptyset = 0$ in this case. In other words, the event function indicates when the corresponding event set is left. We note that an event set overlaps with other event sets, i.e. event sets do not form a partition of $\mathcal{X}$.

The delays from 0 to 90 ms are chosen i.i.d. according to the distribution $\pi = \frac{1}{20} \cdot (0, 2, 2, 4, 4, 3, 2, 1, 1, 1]) \in \mathbb{R}^{10}$ (see Figure 1). This distribution is arbitrary, however, these values (modulo scaling) are inspired by typical delay curves reported by others [10].

In order to compare our new feedback construction from the previous section to the stochastic approach from [7], we compute a feedback trajectory to the inital state $\boldsymbol{x}_0 = [\pi, 0]$ in the original state space $\mathcal{X}$ with an randomly generated sequence of delays.

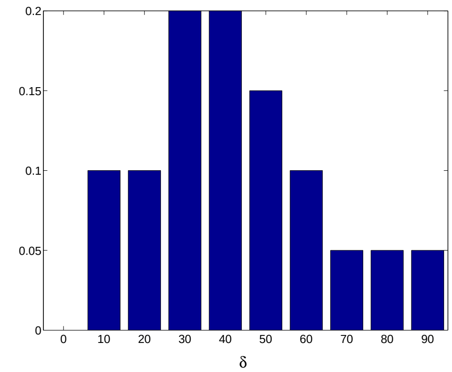

Fig. 1. Distribution of stochastic delays applied in the inverted pendulum example (probability over delay). Delays are discrete from 10 to 90 ms.

Figure 2 shows the trajectory computed with the stochastic approach from [7]. Clearly, the approximate value function is not a (deterministic) Lyapunov function (time instances where the value function increases along the trajectory are marked) and we can only guarantee convergence to the target set $\mathcal{X}^*$ in an expected sense.

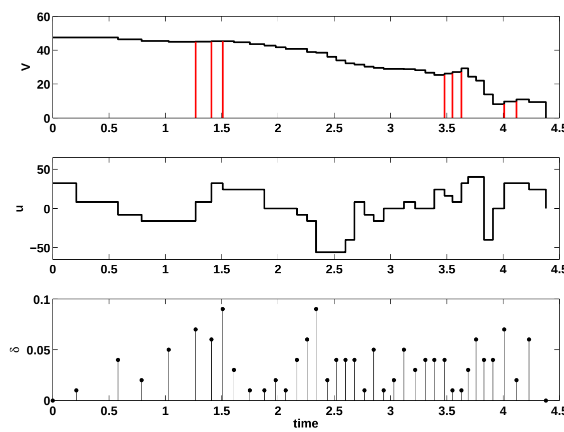

Fig. 2. Value function, control input and delay sequence (from top to bottom) for the feedback trajectory with initial state $\boldsymbol{x}_0 = [\pi, 0]$ using the stochastic feedback from [7].

In contrast to this, for our new feedback construction the value function monotonically decreases (cf. Figure 3), and we can guarantee convergence to the target set $\mathcal{X}^*$.

Finally, in Figure 4 we compare the two trajectories in (the original) phase space. Note the the trajectory using the new feedback reaches the target faster with fewer events.

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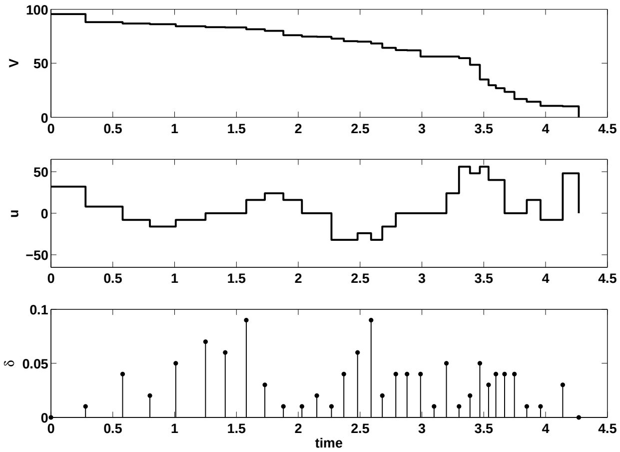

Fig. 3. Value function, control input and delay sequence (from top to bottom) for the feedback trajectory with initial state $\boldsymbol{x}_0 = [\pi, 0]$ using the new feedback construction. Note that the value function decreaes monotonically.

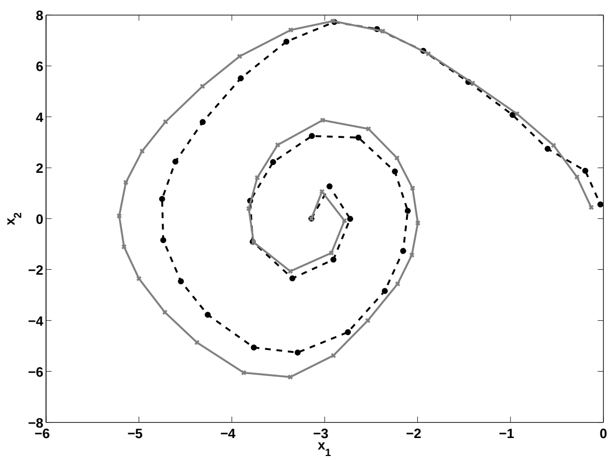

Fig. 4. Feedback trajectories with initial state $\boldsymbol{x}_0 = [\pi, 0]$ in comparison (solid = stochastic feedback from [7], dashed = new feedback).

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