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THE NUMERICAL DETECTION OF CONNECTING ORBITS

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Abstract. We present a new technique for the numerical detection and localization of connecting orbits between hyperbolic invariant sets in parameter dependent dynamical systems. This method is based on set-oriented multilevel methods for the computation of invariant manifolds and it can be applied to systems of moderate dimension. The main idea of the algorithm is to detect intersections of coverings of the stable and unstable manifolds of the invariant sets on different levels of the approximation. We demonstrate the applicability of the new method by three examples.

1. **Introduction.** Today there exist quite sophisticated direct numerical methods for the computation of connecting orbits between steady state solutions in parameter dependent families of dynamical systems (see e.g. [1, 4]). The computation of the connecting orbits is done via the solution of an appropriate boundary value problem. Therefore – as it is the case for all zero finding procedures – a good initial guess as an input for the boundary value problem solver is needed in order to ensure convergence.

In this article we propose a new technique for finding such an initial guess for systems of moderate dimension. In fact, we do not just find an approximation for the parameter value for which there exists a connecting orbit but also a guess for the connecting orbit itself. The underlying idea is to construct coarse coverings of invariant manifolds over a range of parameter values and to search for intersections of these coverings. In the neighborhood of the parameter value for which there exists a connecting orbit there should be nonvanishing intersections even for the case where the coverings are quite tight. This leads to the idea of our *hat algorithm*.

More precisely we cover the invariant manifolds using multilevel subdivision techniques as proposed in [3, 2]. By these methods the invariant manifolds are covered by boxes and the diameter of the boxes is shrinking in the course of the subdivision process. Once a nonvanishing intersection of the box coverings has been found we use the corresponding box cluster as an initial guess for the boundary value problem solver.

By construction the method described in this article is not restricted to the detection of heteroclinic connections between steady state solutions. Rather it can in principle be applied to the detection of connecting orbits between arbitrary invariant sets. However, for simplicity we restrict our considerations to the situation where connecting orbits between different steady state solutions have to be detected.

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An outline of the paper is as follows: in Section 2 we set the scene by a detailed description of the problem. In Section 3 we state the *hat algorithm* and prove that it can indeed be used for the detection of connecting orbits. Finally we illustrate the usefulness of this approach by a couple of numerical examples in Section 4.

2. **The Problem.** We consider a parameter dependent ordinary differential equation

$$\dot{x} = f(x, \lambda), \tag{2.1}$$

where $f : \mathbb{R}^d \times \Lambda \to \mathbb{R}^d$ is a smooth vector field and $\Lambda \subset \mathbb{R}$ is an interval. Denote by x_λ and y_λ ($\lambda \in \Lambda$) – allowing the possibility that $x_\lambda = y_\lambda$ – two one-parameter families of hyperbolic steady state solutions of (2.1). We are interested in the detection of a connecting orbit between two steady states $x_{\bar{\lambda}}$, $y_{\bar{\lambda}}$ while the system parameter λ is varied. In order to ensure that, in principle, connecting orbits can generically occur we assume that

$$\dim(W^u(x_\lambda)) + \dim(W^s(y_\lambda)) = d \quad \text{for all } \lambda \in \Lambda.$$

Here $W^u(x_\lambda)$ and $W^s(y_\lambda)$ denote the unstable resp. the stable manifold of the corresponding steady states.

We do not just aim for a rough guess of the parameter value $\bar{\lambda}$ but also for a guess for the connecting orbit itself. Using these data as initial values one may employ standard techniques on the computation of hetero-/homoclinic orbits, see e.g. [1], [4].

3. **The Detection of Connecting Orbits.** The procedure we are going to describe is based on the set-oriented computation of invariant manifolds for discrete dynamical systems as described in [2] (see also [3]). For convenience we include a brief description of this *continuation algorithm* in Appendix A.

The Underlying Idea. The discrete dynamical system which we are considering is the time-τ map of the flow of (2.1). We approximate this map using an explicit numerical integration scheme and denote by $\mathfrak{U}_j^{(k)}(\lambda)$ and $\mathfrak{S}_j^{(k)}(\lambda)$ the covering of the unstable resp. the stable manifold of x_λ resp. y_λ obtained by the continuation algorithm after k subdivision and j continuation steps. Let

$$\mathfrak{I}_j^{(k)}(\lambda) = \mathfrak{U}_j^{(k)}(\lambda) \cap \mathfrak{S}_j^{(k)}(\lambda).$$

The idea of the algorithm is to find intersections of the box coverings $\mathfrak{U}_j^{(k)}(\lambda)$ and $\mathfrak{S}_j^{(k)}(\lambda)$ for different values of λ and k. Roughly speaking we are going to use the fact that if there exists a connecting orbit for $\lambda = \bar{\lambda}$, then the smaller the distance $|\lambda - \bar{\lambda}|$ the bigger we can choose the number of subdivisions k while still finding a nonempty intersection $\mathfrak{I}_j^{(k)}(\lambda)$. That is, if we plot the maximal k for which a nonempty intersection is found versus λ, we expect to see a schematic picture as illustrated in Figure 1.

The Hat Algorithm. We now describe the algorithm for the detection of connecting orbits in detail. In view of Figure 1 we call it the *hat algorithm.*

Let $\tilde{\Lambda} \subset \Lambda$ be a finite set of parameter values – e.g. a set of equidistant values of λ inside Λ. Denote by $k_{max} \in \mathbb{N}$ the maximal number of subdivisions and by $j_{max} \in \mathbb{N}$ the maximal number of continuation steps performed in the continuation method. Taking $\tilde{\Lambda}$, k_{max} and j_{max} as inputs the hat algorithm computes a function

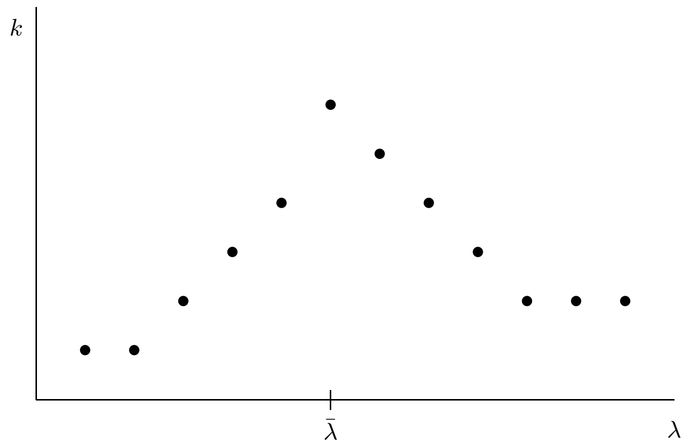


FIGURE 1. Maximal number of subdivisions k for which an intersection of the coverings $\mathfrak{U}_j^{(k)}(\lambda)$ and $\mathfrak{S}_j^{(k)}(\lambda)$ has been found versus the parameter λ (schematic).

$m : \tilde{\Lambda} \to \mathbb{N}$ such that local maximizers of m are close to parameter values $\bar{\lambda}$ for which there may exist a connecting orbit between $x_{\bar{\lambda}}$ and $y_{\bar{\lambda}}$.

More precisely the hat algorithm has the following structure:

Algorithm 1.

m = `hat(`$\tilde{\Lambda}$`,` k_{max}`,` j_{max}`)`
`for all` $\lambda \in \tilde{\Lambda}$
    $k := 0$
    `do`
        $j := 0$
        `do`
            `compute` $\mathfrak{U}_j^{(k)}(\lambda)$ `and` $\mathfrak{S}_j^{(k)}(\lambda)$
            $j := j + 1$
        `while` $\mathfrak{I}_j^{(k)}(\lambda) = \emptyset$ `and` $j < j_{max}$
        $k := k + 1$
    `while` $\mathfrak{I}_j^{(k)}(\lambda) \neq \emptyset$ `and` $k < k_{max}$
    $m(\lambda) = k$
`end`

Remark 1. (i) It is easy to see that the algorithm terminates. In fact, the only reason for the introduction of the parameter k_{max} is to ensure its termination.

(ii) In the case where $x_\lambda = y_\lambda$ one obviously has a nonempty intersection of $\mathfrak{U}_j^{(k)}(\lambda)$ and $\mathfrak{S}_j^{(k)}(\lambda)$ for all $\lambda \in \Lambda$. In this case one needs to modify the intersection test accordingly. In practice this is done by excluding those parts of the box coverings which are inside a neighborhood of the steady state solution x_λ.

We now show that the hat algorithm can indeed be used for the detection of nondegenerate heteroclinic codimension one bifurcations (see [5]). Let ϕ^t denote the flow of the system (2.1). Then we define for an equilibrium p and $T \geq 0$

$$W_T^u(p) = \bigcup_{0\leq t\leq T} \phi^t(W_{\rm loc}^u(p)), \qquad W_T^s(p) = \bigcup_{-T\leq t\leq 0} \phi^t(W_{\rm loc}^s(p)),$$

where $W_{\rm loc}^{u,s}(p)$ are the local (un)stable manifolds of p. If there exists an orbit connecting x_λ and y_λ for $\lambda = \bar\lambda$ then there is a $T \geq 0$ such that

$$W_T^u(x_{\bar\lambda}) \cap W_T^s(y_{\bar\lambda}) \neq \emptyset.$$

(In practice the minimal T with this property is unkown and this is the reason why one should choose a rather large j_{max} in the realization of the hat algorithm.) Accordingly the intersection $\mathfrak{I}_j^{(k)}(\bar\lambda)$ will be nonempty for all box coverings $\mathfrak{U}_j^{(k)}(\bar\lambda)$ and $\mathfrak{S}_j^{(k)}(\bar\lambda)$ satisfying

$$W_T^u(x_{\bar\lambda}) \subset \bigcup_{B\in\mathfrak{U}_j^{(k)}(\bar\lambda)} B \quad \text{and} \quad W_T^s(x_{\bar\lambda}) \subset \bigcup_{B\in\mathfrak{S}_j^{(k)}(\bar\lambda)} B.$$

In fact since there exists a connecting orbit for $\lambda = \bar\lambda$, $\mathfrak{I}_j^{(k)}(\bar\lambda)$ will be nonempty for all k if j is big enough. But also the converse is true:

Proposition 1. *If for some $\bar\lambda \in \tilde\Lambda$ and $j \in \mathbb{N}$ the intersection $\mathfrak{I}_j^{(k)}(\bar\lambda)$ is nonempty for all k, then there exists an orbit of (2.1) connecting $x_{\bar\lambda}$ and $y_{\bar\lambda}$.*

Proof. By assumption there is a j such that

$$\mathfrak{I}_j^{(k)}(\bar\lambda) = \mathfrak{U}_j^{(k)}(\bar\lambda) \cap \mathfrak{S}_j^{(k)}(\bar\lambda) \neq \emptyset$$

for all k. Both $\mathfrak{U}_j^{(k)}(\bar\lambda)$ and $\mathfrak{S}_j^{(k)}(\bar\lambda)$ form nested sequences of compact sets which converge to part of the unstable (stable) manifold of $x_{\bar\lambda}$ ($y_{\bar\lambda}$). Therefore

$$\lim_{k\to\infty} \mathfrak{I}_j^{(k)}(\bar\lambda) \neq \emptyset$$

which proves the proposition.

□

For the statement of the following result it is convenient to introduce a specific choice for the set $\tilde\Lambda$. If $\Lambda = [a, b]$ then we define for $n \in \mathbb{N}$

$$h = \frac{b-a}{n} \quad \text{and} \quad \tilde\Lambda_h = \{a + ih : i = 0, 1, \ldots, n\}.$$

Proposition 2. *Suppose that for some $\bar\lambda \in \Lambda$ the system (2.1) undergoes a non-degenerate heteroclinic codimension one bifurcation with respect to the steady state solutions x_λ and y_λ. Then for each integer $k_{max} > 0$ there are $h > 0$ and $j_{max} > 0$ such that those $\tilde\lambda \in \tilde\Lambda_h$ for which $|\tilde\lambda - \bar\lambda|$ is minimal satisfy $m(\tilde\lambda) = k_{max}$. These values are in particular local maximizers of $m : \tilde\Lambda \to \mathbb{N}$. (Here m denotes the quantity computed by the hat algorithm.)*

Proof. Suppose that j_{max} is chosen in such a way that $\mathfrak{I}_j^{(k)}(\bar\lambda)$ is nonempty for all k and $j = j_{max} - 1$. Then, by construction of the hat algorithm, $m(\bar\lambda) = k_{max}$. Since $W_T^u(x_\lambda)$ and $W_T^s(y_\lambda)$ depend continuously on λ we can conclude that there is an $\eta > 0$ such that $m(\lambda) = k_{max}$ for all $\lambda \in (\bar\lambda - \eta, \bar\lambda + \eta)$. Now choose $h > 0$ so small that $\tilde\Lambda_h \cap (\bar\lambda - \eta, \bar\lambda + \eta) \neq \emptyset$.

□

4. **Numerical Examples.** We demonstrate the performance of the hat algorithm by three examples. In addition to the figures provided here illustrating videos can be downloaded from the homepage of the authors. For details on the implementation of the continuation method see [3, 2].

Example 1. Consider the system

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -\theta\, y + x^3 + x^2 - 3\, x + 1. \end{aligned} \tag{4.1}$$

Our aim is to detect an orbit connecting the equilibrium $x_\lambda = (-1 - \sqrt{2}, 0)$ with the equilibrium $y_\lambda = (-1 - \sqrt{2}, 0)$. We use the time-0.2-map of the corresponding flow and choose $\tilde{\Lambda} = \{0.2, 0.4, 0.6, \ldots, 2.8, 3.0\}$, as well as $k_{max} = 26$ and $j_{max} = 20$ as inputs for the hat algorithm. Figure 2 shows the graph of the computed function m. It indicates that for $\theta \approx 1.6$ there indeed exists a connecting orbit.

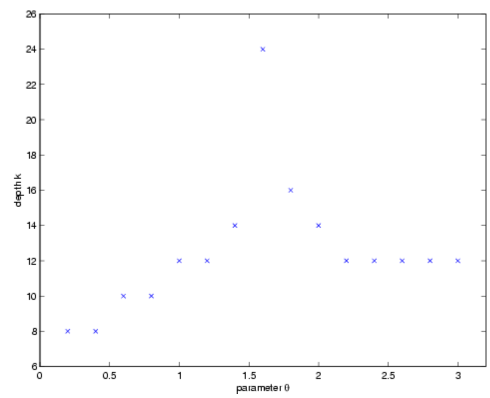


FIGURE 2. The function m for Example 1.

In order to illustrate the behavior of the hat algorithm we additionally show the box coverings obtained

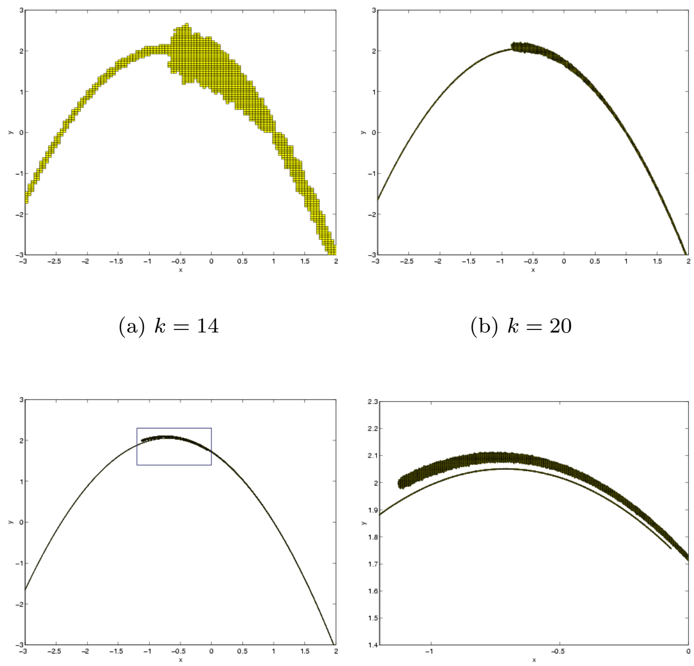
(i) for a fixed parameter value $\theta = 1.6$ and for different numbers of subdivisions k (Figure 3);
(ii) for a fixed depth $k = 14$ and different parameter values θ (Figure 4).

Example 2. In the second example we aim for the computation of a homoclinic orbit. Consider

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= x - x^2 + \lambda y + 0.5xy. \end{aligned} \tag{4.2}$$

In [1] a homoclinic orbit is computed which connects the steady state $(0, 0)$ with itself for $\lambda \approx -0.4295$. Considering the time-1-map of the flow of (4.2) we apply the hat algorithm with $\tilde{\Lambda} = \{-1.4, -1.3, -1.2, \ldots, 0.3, 0.4\}$, $k_{max} = 16$ and $j_{max} = 10$. As the outer box for the continuation algorithm we choose $Q = [-2, 2] \times [-2, 2]$. In Figure 5 we show the graph of the resulting function m. The result indeed indicates that there exists a homoclinic orbit for the origin near $\lambda = -0.4$.

Again we show the box coverings obtained for a fixed parameter value $\lambda = -0.4$ and for different numbers of subdivisions k (Figure 6), as well as for a fixed depth $k = 14$ and different parameter values λ (Figure 7).



(a) $k = 14$ (b) $k = 20$

(c) $k = 26$ (d) Zoom in (c)

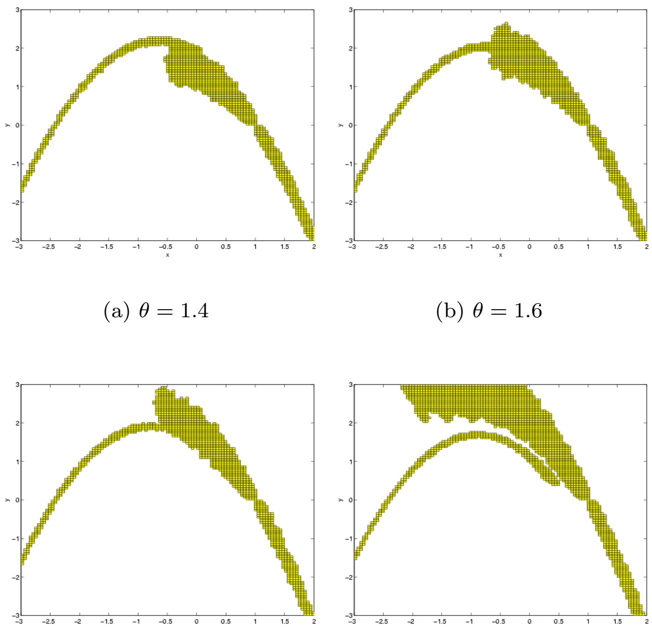
FIGURE 3. Example 1: The coverings $\mathfrak{U}^{(k)}(1.6)$ and $\mathfrak{S}^{(k)}(1.6)$ for different k.

Example 3. Let us finally consider a non-planar differential equation, the celebrated *Lorenz-system*

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= \rho x - y - xz \\ \dot{z} &= xy - \beta z \end{aligned}$$

with parameter values $\sigma = 10$ and $\beta = \frac{8}{3}$. It is well known that there exists a homoclinic orbit for the origin near $\rho = 13.93$, see e.g. [6]. We consider the time-T map of the corresponding flow with $T = 0.2$ and apply the hat algorithm with $\tilde{\Lambda} = \{7, 8, 9, \ldots, 19, 20\}$, $k_{max} = 30$ and $j_{max} = 8$. In Figure 8(a) we show the result of this computation. Additionally in Figure 8(b) we show the number of boxes in the intersection of $\mathfrak{U}^{(27)}(\rho)$ and $\mathfrak{S}^{(27)}(\rho)$ for $\rho = 13, 13.5, 14, 14.5, 15$. Again we illustrate the computations by plotting several computed coverings, see Figures 9 and 10.

Appendix A. **The continuation algorithm.** In this section we briefly outline the continuation method for the set-oriented approximation of stable/unstable manifolds. See [3, 2] for a more detailed exposition. For simplicity we consider the case of the computation of the unstable manifold $W^u(p)$ of a hyperbolic fixed point p. The method can be extended in an obvious manner to approximate invariant manifolds of general hyperbolic invariant sets.



(a) $\theta = 1.4$ (b) $\theta = 1.6$

(c) $\theta = 1.8$ (d) $\theta = 2.2$

FIGURE 4. Example 1: The coverings $\mathfrak{U}^{(14)}(\theta)$ and $\mathfrak{S}^{(14)}(\theta)$ in dependence of θ.

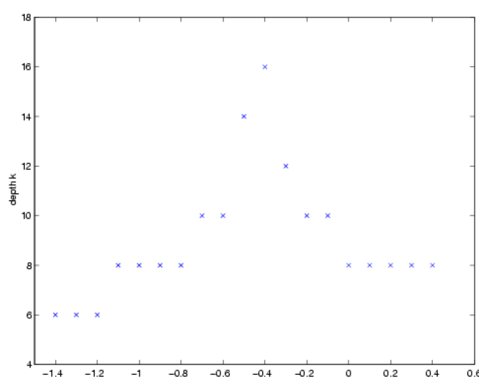
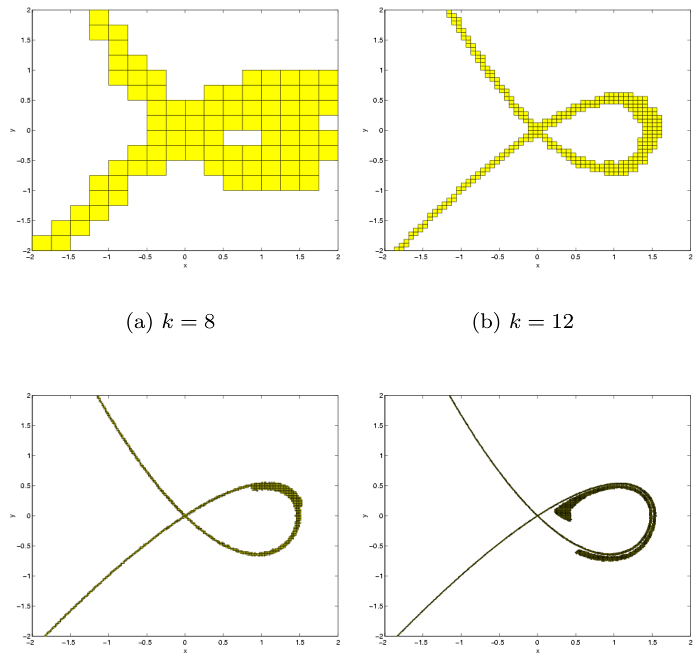


FIGURE 5. The function m for Example 2.

Recall that we consider a (time-) discrete dynamical system

$$x_{n+1} = f(x_n), \quad n = 0, 1, \ldots,$$

where $f : \mathbb{R}^d \to \mathbb{R}^d$ is a diffeomorphism. Let $Q \subset \mathbb{R}^d$ be a compact set containing p in which we want to approximate part of $W^u(p)$. A *partition* $\mathfrak{P}$ of Q consists of



(a) $k = 8$ (b) $k = 12$

(c) $k = 16$ (d) $k = 18$

FIGURE 6. Example 2: The coverings $\mathfrak{U}^{(k)}(-0.4)$ and $\mathfrak{S}^{(k)}(-0.4)$ for different values of k.

finitely many subsets of Q such that

$$\bigcup_{B\in\mathfrak{P}} B = Q \quad \text{and} \quad B \cap B' = \emptyset \quad \text{for all } B, B' \in \mathfrak{P},\ B \neq B'.$$

Let $\mathfrak{P}_\ell$, $\ell \in \mathbb{N}$, be a nested sequence of successively finer partitions of Q, requiring that for all $B \in \mathfrak{P}_\ell$ there exist $B_1, \ldots, B_m \in \mathfrak{P}_{\ell+1}$ such that $B = \bigcup_i B_i$ and $\operatorname{diam}(B_i) \leq \theta \operatorname{diam}(B)$ for some $0 < \theta < 1$.

Let $C \in \mathfrak{P}_\ell$ be a neighborhood of the hyperbolic fixed point p. Then an application of the subdivision algorithm from [3] to this neighborhood allows to produce a tight box covering of the set

$$A_C = W^u_{\text{loc}}(p) \cap C.$$

More precisely, applying the subdivision procedure k times to $\mathcal{B}_0 = \{C\}$, we obtain a covering $\mathcal{B}_k \subset \mathfrak{P}_{\ell+k}$ of the local unstable manifold $W^u_{\text{loc}}(p) \cap C$, that is,

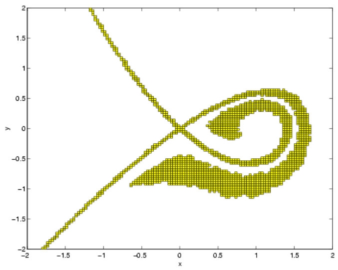
$$A_C = W^u_{\text{loc}}(p) \cap C \subset \bigcup_{B\in\mathcal{B}_k} B.$$

This covering converges to $W^u_{\text{loc}}(p) \cap C$ for $k \to \infty$, see again [3].

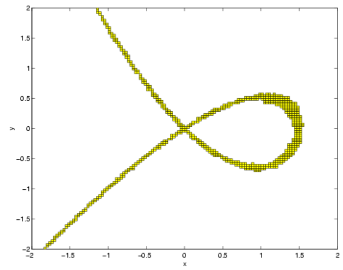
The continuation algorithm now works as follows. For a fixed k we define a sequence $\mathfrak{C}^{(k)}_0, \mathfrak{C}^{(k)}_1, \ldots$ of subsets $\mathfrak{C}^{(k)}_j \subset \mathfrak{P}_{\ell+k}$ by

(i) *Initialization:*

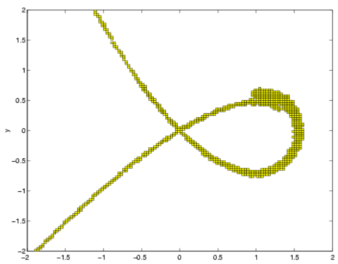
$$\mathfrak{C}^{(k)}_0 = \mathcal{B}_k.$$



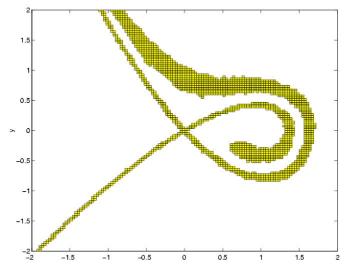
(a) $\lambda = -0.3$



(b) $\lambda = -0.4$

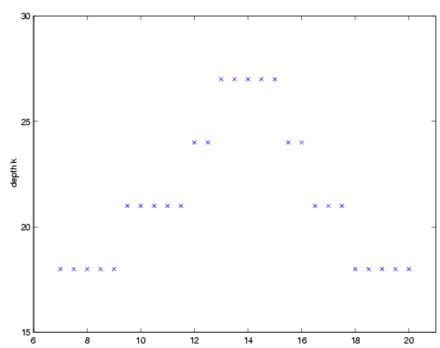


(c) $\lambda = -0.5$

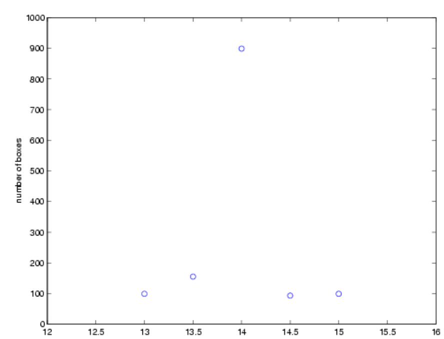


(d) $\lambda = -0.6$

FIGURE 7. Example 2: The coverings $\mathfrak{U}^{(14)}(\lambda)$ and $\mathfrak{S}^{(14)}(\lambda)$ in dependence of λ.



(a) The function m.

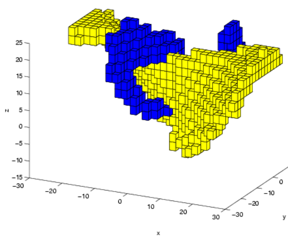


(b) The number of boxes in $\mathfrak{I}^{(27)}_{j_{max}}(\rho)$ as a function of λ

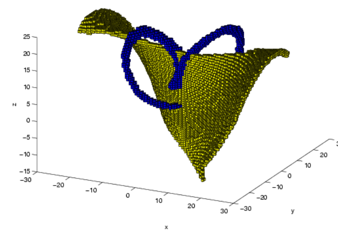
FIGURE 8. Results for Example 3.

(ii) *Continuation:* For $j = 0, 1, 2, \dots$ define

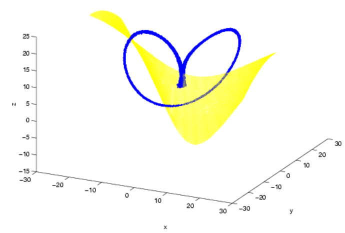
$$\mathfrak{C}^{(k)}_{j+1} = \left\{ B \in \mathfrak{P}_{\ell+k} \ : \ B \cap f(B') \neq \emptyset \text{ for some } B' \in \mathfrak{C}^{(k)}_j \right\}.$$



(a) $k = 15$



(b) $k = 21$



(c) $k = 27$

FIGURE 9. Example 3: The coverings $\mathfrak{U}^{(k)}(-14)$ and $\mathfrak{S}^{(k)}(-14)$ for different k.

It can be shown that this continuation procedure indeed converges to a compact subset of the unstable manifold, see [2].

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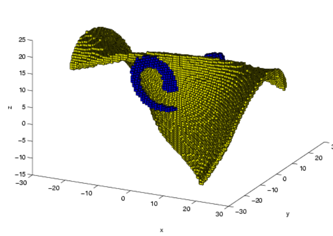
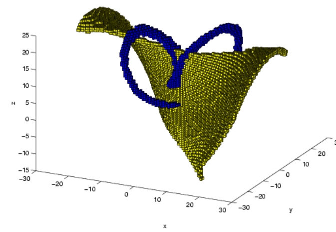
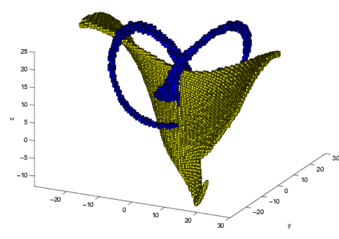


(a) $\rho = 10$



(b) $\rho = 14$



(c) $\rho = 17$

FIGURE 10. Example 3: The coverings $\mathfrak{U}^{(21)}(\rho)$ and $\mathfrak{S}^{(21)}(\rho)$ in dependence of ρ.