A NOTE ON THE EXPANSION OF THE SMALLEST EIGENVALUE DISTRIBUTION OF THE LUE AT THE HARD EDGE[1]

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In a recent paper, Edelman, Guionnet and Péché conjectured a particular n^{-1} correction term of the smallest eigenvalue distribution of the Laguerre unitary ensemble (LUE) of order n in the hard-edge scaling limit: specifically, the derivative of the limit distribution, that is, the density, shows up in that correction term. We give a short proof by modifying the hard-edge scaling to achieve an optimal $O(n^{-2})$ rate of convergence of the smallest eigenvalue distribution. The appearance of the derivative follows then by a Taylor expansion of the less optimal, standard hard-edge scaling. We relate the n^{-1} correction term further to the logarithmic derivative of the Bessel kernel Fredholm determinant in the work of Tracy and Widom.

1. Introduction. We recall from random matrix theory that the smallest eigenvalue distribution of the LUE of order n with parameter $a > -1$ is given by the Fredholm determinant [6], Chapter 9,

$$\mathbb{P}(\lambda_{\min} \geq s) = \det\big(I - K_n^a|_{L^2(0,s)}\big),$$

induced by the Laguerre projection kernel

$$K_n^a(x, y) = \sum_{k=0}^{n-1} \phi_k^a(x)\phi_k^a(y), \qquad \phi_k^a(x) = \sqrt{\frac{k!}{\Gamma(k+a+1)}} e^{-x/2} x^{a/2} L_k^a(x).$$

Here, L_k^a denotes the generalized Laguerre polynomial of degree k. By Christoffel–Darboux and the relation $L_n^{a-1}(x) + L_{n-1}^a(x) = L_n^a(x)$ (cf. [1], equation (22.7.30)), one gets the closed form

$$K_n^a(x, y) = \frac{n! e^{-(x+y)/2}(xy)^{a/2}}{\Gamma(n+a)} \cdot \frac{L_n^a(x)L_n^{a-1}(y) - L_n^{a-1}(x)L_n^a(y)}{x - y}.$$

Using the Mehler–Heine type asymptotics ([10], Theorem 8.1.3) of the Laguerre polynomials, which holds uniformly for bounded z in the complex plane,[2]

$$n^{-a} L_n^a(z/n) = z^{-a/2} J_a(2\sqrt{z}) + o(1) \qquad (n \to \infty)$$

one immediately obtains, as first done by Forrester ([5], equation (2.6); see also [6], Section 7.2.1), that in the hard-edge scaling

$$X = \frac{x}{4n},$$

likewise for Y and y, there is the limit, locally uniform for positive x and y,

$$K_n^a(X, Y)\, dX = \left(K_\infty^a(x, y) + o(1) \right) dx \qquad (n \to \infty)$$

with the Bessel kernel

$$K_\infty^a(x, y) = \frac{\sqrt{y}\, J_a(\sqrt{x}) J_{a-1}(\sqrt{y}) - \sqrt{x}\, J_{a-1}(\sqrt{x}) J_a(\sqrt{y})}{2(x - y)}$$

$$= \frac{\sqrt{y}\, J_a(\sqrt{x}) J_a'(\sqrt{y}) - \sqrt{x}\, J_a'(\sqrt{x}) J_a(\sqrt{y})}{2(x - y)}.$$

Lifted to the convergence of the induced Fredholm determinants (see the next section for details), one thus gets the hard-edge scaling limit of the LUE as a limit of distributions, namely as $n \to \infty$

$$(1) \qquad F_n^a(s) = \mathbb{P}\left(\lambda_{\min} \geq \frac{s}{4n} \right) \to F_\infty^a(s) = \det\left(I - K_\infty^a|_{L^2(0,s)} \right).$$

Based on an identity of finite-dimensional Bessel function determinants obtained from symbolical and numerical computer experiments, Edelman, Guionnet and Péché [4], page 14, *conjectured* the following refinement of (1):

$$(2) \qquad F_n^a(s) = F_\infty^a(s) + \frac{a}{2n} s f_\infty^a(s) + O(n^{-2}), \qquad f_\infty^a(s) = \frac{d}{ds} F_\infty^a(s).$$

In this note, we will give a short proof that this expansion, in fact, holds true.

REMARK. At FoCM'14, Grégory Schehr [9] presented yet another proof (joint work with Anthony Perret) of this expansion which he had obtained as a byproduct of a new approach to the Painlevé III representation [11] of the Bessel kernel determinant.

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[2]Hence, for all $a \in \mathbb{R}$,

$$z^{-a/2} J_a(2\sqrt{z}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(a + k + 1)} z^k$$

constitutes a uniquely defined *entire* function of the complex variable z.

2. A short proof of the expansion.   Lemma 4.1 of [4] easily implies a refinement of the Mehler–Heine type asymptotics in the form of an expansion (see also [8], page 29, for $a = 0$ and [12], page 156, for the general case), namely,

(3)
$$(n + a)^{-a} L_n^a(z/(n + a))$$

$$= z^{-a/2} J_a(2\sqrt{z}) - \frac{1}{2n} z^{-(a-2)/2} J_{a-2}(2\sqrt{z}) + O(n^{-2}),$$

uniformly for bounded complex z. Subject to the following modified hard-edge scaling, likewise for Y and y,

$$X = \frac{x}{4n}\left(1 - \frac{a + c}{2n}\right),$$

which transforms the Laguerre kernel according to

$$K_n^a(X, Y)\, dX = \tilde{K}_n^a(x, y)\, dx,$$

this gives after some routine calculation[3] the expansion

(4)
$$(xy)^{-a/2} \tilde{K}_n^a(x, y)$$

$$= (xy)^{-a/2} K_\infty^a(x, y) - \frac{c}{8n} x^{-a/2} J_a(\sqrt{x}) y^{-a/2} J_a(\sqrt{y}) + O(n^{-2}),$$

uniformly for bounded complex x and y; in particular, uniformly for $x, y \in [0, s]$. Because of the pre-factor $(xy)^{-a/2}$, all the terms in (4) represent *entire* kernels; see the footnote in the last section. Now, such a pre-multiplication of a kernel K by $(xy)^{-a/2}$ leaves its Fredholm determinant invariant if we transform the measure defining the underlying L^2-space accordingly:

$$\det(I - K|_{L^2(0,s)}) = \det(I - (xy)^{-a/2} K|_{L^2((0,s);\nu_a)}), \qquad \nu_a(dx) = x^a\, dx,$$

which follows simply from conjugating K with the *unitary* transformation

$$U : L^2(0, s) \to L^2((0, s), \nu_a), \qquad f(x) \mapsto x^{-a/2} f(x).$$

Because of $a > -1$, the transformed measure ν_a on $[0, s]$ has finite mass. Since the Fredholm determinant, when defined with respect to a measure ν of finite mass, is locally Lipschitz continuous with respect to the *uniform* convergence of the kernels (see [2], Lemma 3.4.5), the *uniform* kernel expansion (4) on $(0, s)^2$ immediately lifts to an expansion of the induced Fredholm determinants. For the particular choice $c = 0$, which eliminates the $O(n^{-1})$ term, we thus get

(5)
$$F_n^a\left(\left(1 - \frac{a}{2n}\right)s\right) = F_\infty^a(s) + O(n^{-2}).$$

Now, a simple Taylor expansion readily establishes (2):

$$F_n^a(s) = F_\infty^a\left(\left(1 - \frac{a}{2n}\right)^{-1} s\right) + O(n^{-2}) = F_\infty^a(s) + \frac{a}{2n} s f_\infty^a(s) + O(n^{-2}).$$

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[3] A *Mathematica* notebook checking this result can be found at arXiv:1504.00235.

REMARK. The choice $c = -a$ of the additional scaling parameter which is implicitly used in (2) is not the best possible one; the optimally modified hard-edge scaling is given by (5). Obtaining such a second-order convergence rate by appropriately modifying the scaling was stimulated by the corresponding work of Johnstone and Ma [7] for the *largest* eigenvalue distributions of the Gaussian unitary ensemble (GUE) at the soft edge; cf. also the work of Choup [3] on the largest eigenvalue distributions of GUE and LUE.

3. Relation to the Tracy–Widom theory. Writing $\phi_a(x) = J_a(\sqrt{x})$ for short and, as integral operators acting on $L^2(0, s)$,

$$\hat{K}_n^a = K_\infty^a + \frac{a}{8n}\phi_a \otimes \phi_a,$$

$$I - \hat{K}_n^a = (I - K_\infty^a)\left(I - \frac{a}{8n}(I - K_n^a)^{-1}\phi_a \otimes \phi_a\right),$$

we get from (4) with $c = -a$, by the same reasoning as in the last section,

$$F_n^a(s) = \det(I - \hat{K}_n^a) + O(n^{-2})$$

$$= \det(I - K_\infty^a) \cdot \det\left(I - \frac{a}{8n}(I - K_n^a)^{-1}\phi_a \otimes \phi_a\right) + O(n^{-2})$$

$$= F_\infty^a(s) \cdot \left(1 - \frac{a}{8n}\langle(I - K_n^a)^{-1}\phi_a, \phi_a\rangle_{L^2(0,s)}\right) + O(n^{-2}).$$

Comparing the n^{-1} term of the expansion with (2) establishes the relation

$$(6) \qquad -\frac{1}{4}\langle(I - K_n^a)^{-1}\phi_a, \phi_a\rangle_{L^2(0,s)} = s\frac{f_\infty^a(s)}{F_\infty^a(s)} = s\frac{d}{ds}\log F_\infty^a(s),$$

which can already be found in the work of Tracy and Widom. In fact, from the relation $\log\det = \operatorname{tr}\log$, one gets that the logarithmic derivative of a Fredholm determinant $F(s)$ of a trace class operator K acting on $L^2(0, s)$ is generally given by [11], equation (1.5),

$$\frac{d}{ds}F(s) = \frac{d}{ds}\log\det(I - K|_{L^2(0,s)}) = -R(s, s),$$

where $R(x, y)$ is the kernel of the operator $K(I - K)^{-1}$. Specifically, in the case of the Bessel kernel, Tracy and Widom calculated [11], equations (2.5) and (2.21), that

$$sR(s, s) = \tfrac{1}{4}\langle(I - K_n^a)^{-1}\phi_a, \phi_a\rangle_{L^2(0,s)},$$

which finally reproves (6).

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