

Joint distribution of the first and second eigenvalues at the soft edge of unitary ensembles

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Abstract

The density function for the joint distribution of the first and second eigenvalues at the soft edge of unitary ensembles is found in terms of a Painlevé II transcendent and its associated isomonodromic system. As a corollary, the density function for the spacing between these two eigenvalues is similarly characterized. The particular solution of Painlevé II that arises is a double shifted Bäcklund transformation of the Hastings–McLeod solution, which applies in the case of the distribution of the largest eigenvalue at the soft edge. Our deductions are made by employing the hard-to-soft edge transition, involving the limit as the repulsion strength at the hard edge $a \rightarrow \infty$, to existing results for the joint distribution of the first and second eigenvalue at the hard edge (Forrester and Witte 2007 *Kyushu J. Math.* **61** 457–526). In addition recursions under $a \mapsto a + 1$ of quantities specifying the latter are obtained. A Fredholm determinant type characterization is used to provide accurate numerics for the distribution of the spacing between the two largest eigenvalues.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Fundamental to random matrix theory and its applications is the soft edge scaling limit of unitary invariant ensembles. As a concrete example, consider the Gaussian unitary ensemble, specified by the measure on complex Hermitian matrices H proportional to $\exp(-\text{Tr } H^2)(dH)$. This measure is unchanged by the mapping $H \mapsto UHU^\dagger$, for U unitary, and is thus a unitary

invariant. To leading order the support of the spectrum is $(-\sqrt{2N}, \sqrt{2N})$, although there is a non-zero probability of eigenvalues in $(-\infty, -\sqrt{2N}) \cup (\sqrt{2N}, \infty)$, and for this reason the neighbourhood of $\sqrt{2N}$ (or $-\sqrt{2N}$) is referred to as the soft edge. Moreover, upon the scaling of the eigenvalues $\lambda_\ell \mapsto \sqrt{2N} + X_\ell/\sqrt{2N}^{1/6}$, the mean spacing between eigenvalues in the neighbourhood of the largest eigenvalue is of order unity. Taking the $N \rightarrow \infty$ limit with this scaling gives a well-defined statistical mechanical state, which is an example of a determinantal point process, and defined in terms of its k -point correlation functions by

$$\rho_{(k)}^{\text{soft}}(x_1, \dots, x_k) = \det [K^{\text{soft}}(x_j, x_\ell)]_{j,\ell=1,\dots,k}, \tag{1.1}$$

where K^{soft} —referred to as the correlation kernel—is given in terms of Airy functions by

$$K^{\text{soft}}(x, y) := \frac{\text{Ai}(x)\text{Ai}'(y) - \text{Ai}(y)\text{Ai}'(x)}{x - y}. \tag{1.2}$$

The determinantal form (1.1) implies that in the soft edge scaled state, the probability of there being no eigenvalues in the interval (s, ∞) , is given by [10]

$$\begin{aligned} E_2^{\text{soft}}(0; (s, \infty)) &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_s^{\infty} dx_1 \dots \int_s^{\infty} dx_k \rho_{(k)}^{\text{soft}}(x_1, \dots, x_k), \\ &= \det (1 - \mathbb{K}_{(s,\infty)}^{\text{soft}}), \end{aligned} \tag{1.3}$$

where $\mathbb{K}_{(s,\infty)}^{\text{soft}}$ is the integral operator on (s, ∞) with kernel $K^{\text{soft}}(x, y)$ (as given in (1.2)). The first equality in (1.3) is generally true for a one-dimensional point process, while the second equality follows from the Fredholm theory [32] (see also the comments following (1.7)). The structure of the kernel (1.2) makes it of a class referred to as integrable [17], and generally this class of integrable kernels have intimate connections to integrable systems. Indeed one has that [30]

$$\det (1 - \mathbb{K}_{(s,\infty)}^{\text{soft}}) = \exp \left(- \int_s^{\infty} (t - s)q^2(t) dt \right), \tag{1.4}$$

where $q(t)$ satisfies the particular Painlevé II ordinary differential equation (ODE) ($\dot{} \equiv d/dt$)

$$\ddot{q} = 2q^3 + tq, \tag{1.5}$$

subject to the boundary condition

$$q(t) \underset{t \rightarrow \infty}{\sim} \text{Ai}(t). \tag{1.6}$$

Our interest in this paper is in the joint distribution of the largest and second largest eigenvalue at the soft edge, and the corresponding distribution of the spacing between them. Let $p_{(2)}^{\text{soft}}(x_1, x_2)$, $x_1 > x_2$, denote the density function of the joint distribution. Then analogous to the first equality in (1.3) we have

$$\begin{aligned} p_{(2)}^{\text{soft}}(x_1, x_2) &= \det \begin{bmatrix} K^{\text{soft}}(x_1, x_1) & K^{\text{soft}}(x_1, x_2) \\ K^{\text{soft}}(x_2, x_1) & K^{\text{soft}}(x_2, x_2) \end{bmatrix} + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{x_2}^{\infty} dy_1 \dots \int_{x_2}^{\infty} dy_k \\ &\times \det \begin{bmatrix} K^{\text{soft}}(x_1, x_1) & K^{\text{soft}}(x_1, x_2) & [K^{\text{soft}}(x_1, y_\ell)]_{\ell=1,\dots,k} \\ K^{\text{soft}}(x_2, x_1) & K^{\text{soft}}(x_2, x_2) & [K^{\text{soft}}(x_2, y_\ell)]_{\ell=1,\dots,k} \\ [K^{\text{soft}}(y_j, x_1)]_{j=1,\dots,k} & [K^{\text{soft}}(y_j, x_2)]_{j=1,\dots,k} & [K^{\text{soft}}(y_j, y_\ell)]_{j,\ell=1,\dots,k} \end{bmatrix}. \end{aligned} \tag{1.7}$$

This equality can be established by generalizing the methods employed in proposition 5.1.2, the results given in exercise 5.1, q. 3 and the definitions in proposition 8.1.2 of [11]. With

$A^{\text{soft}}(s)$ denoting the density function for the spacing between the two largest eigenvalues we have

$$A^{\text{soft}}(s) = \int_{-\infty}^{\infty} dx p_{(2)}^{\text{soft}}(x + s, x). \tag{1.8}$$

We seek to characterize (1.7) and (1.8) in a form analogous to (1.4). This involves functions which are components of a solution of a particular isomonodromic problem relating to the PII equation. Such characterizations have appeared in other problems in random matrix theory and related growth processes [1, 5, 14, 28]. Our approach stands in contrast to the work of Tracy and Widom [30] where recurrence relations are given for the separate distributions of the largest and next-largest eigenvalues at the soft edge involving the generating function

$$D_2^{\text{soft}}(0; (s, \infty); \xi) = \det(1 - \xi \mathbb{K}_{(s, \infty)}^{\text{soft}}). \tag{1.9}$$

The reconciliation of these approaches remains an open problem even though the latter theory can employ a generalization of (1.5) now subject to the boundary condition $q(t) \sim_{t \rightarrow \infty} \sqrt{\xi} \text{Ai}(t)$ [7].

The starting point for us is our earlier study [14] specifying the joint distribution of the first and second smallest eigenvalues, and the corresponding spacing distribution between these eigenvalues, at the hard edge of unitary ensembles. In random matrix theory the latter applies when the eigenvalue density is strictly zero on one side of its support, and is specified by the determinantal point process with correlation kernel

$$K^{\text{hard}, a}(x, y) = \frac{\sqrt{y} J_a(\sqrt{x}) J'_a(\sqrt{y}) - \sqrt{x} J'_a(\sqrt{x}) J_a(\sqrt{y})}{2(x - y)}, \tag{1.10}$$

where $x, y > 0$, and $J_a(x)$ and $J'_a(x)$ are the standard Bessel function of the first kind and its derivative, respectively, see section 10.2(ii) of [26]. Note the dependence on the parameter a ($a > -1$) which physically represents a repulsion from the origin. The relevance to the study of the soft edge is that upon the scaling

$$x \mapsto a^2[1 - 2^{2/3} a^{-2/3} x], \tag{1.11}$$

(and similarly y), as $a \rightarrow \infty$ the hard edge kernel (1.10) limits to the soft edge kernel, and consequently the hard edge state as defined by its correlation functions limits to the soft edge state [4]. Thus our task is to compute this limit in the expressions from [14]. Moreover, recurrences under the mapping $a \mapsto a + 1$ of all quantities specifying the joint distribution at the hard edge will be given.

Explicitly, let $q(t; \alpha) =: q_\alpha(t)$ satisfy the standard form of the second Painlevé equation

$$\ddot{q} = 2q^3 + tq + \alpha, \tag{1.12}$$

with $p = \dot{q} + q^2 + \frac{1}{2}t$. In our application we have the specialization $\alpha = \frac{3}{2}$. Furthermore introduce $U(x; t), V(x; t)$ through the Lax pair equations

$$\partial_x \begin{pmatrix} U \\ V \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & 0 \end{pmatrix} x + \begin{pmatrix} -q - \frac{2}{p} & -1 \\ \frac{1}{2}(t - p) + [q + \frac{2}{p}]^2 & q + \frac{2}{p} \end{pmatrix} + \begin{pmatrix} 1 & p \\ 0 & -1 \end{pmatrix} \frac{1}{x} \right\} \begin{pmatrix} U \\ V \end{pmatrix}, \tag{1.13}$$

and

$$\partial_t \begin{pmatrix} U \\ V \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 1 \\ 0 & -2[q + \frac{2}{p}] \end{pmatrix} \right\} \begin{pmatrix} U \\ V \end{pmatrix}. \tag{1.14}$$

With this notation, we show in propositions 9 and 10 below, that subject to some specific boundary conditions for the transcendents and isomonodromic components involved

$$\begin{aligned}
 p_{(2)}^{\text{soft}}(t, t-x) &= \frac{1}{4\pi} p_{(1)}^{\text{soft}}(t) t^{-5/2} \\
 &\times \exp\left(-\frac{4}{3}t^{3/2}\right) \exp\left(\int_{2^{1/3}t}^{\infty} dy \left\{ \left(2q + \frac{4}{p}\right)(-y) - \sqrt{2y} - \frac{5}{2y} \right\}\right) \\
 &\times (U\partial_x V - V\partial_x U)(-2^{1/3}x; -2^{1/3}t).
 \end{aligned} \tag{1.15}$$

In section 2 the evaluation of the joint distribution of the first and second eigenvalue at the hard edge from [14] is reviewed. This involves quantities relating to the Hamiltonian formulation of the Painlevé III' equation, and to an isomonodromic problem for the generic Painlevé III' equation. Details of these aspects are discussed in separate subsections, with special emphasis placed on the transformation of the relevant quantities under the mapping $a \mapsto a+1$. Second order recurrences are obtained. In section 2.4 initial conditions for these recurrences are specified. Section 3 is devoted to the computation of the hard-to-soft edge scaling of the quantities occurring in the evaluation of the joint distribution of the first and second eigenvalue at the hard edge. This allows us to evaluate the joint distribution of the first and second eigenvalues at the soft edge in terms of a Painlevé II transcendent and its associated isomonodromic system. In section 4 we make use of a Fredholm determinant interpretation of (1.7) to give accurate numerics for the spacing density function $A^{\text{soft}}(s)$ as specified by (1.8).

2. Hard edge $a > 0$ joint distribution of the first and second eigenvalues

2.1. The result from [14]

Let $p_{(2)}^{\text{hard}, a}(x_1, x_2)$, $x_2 > x_1$ denote the joint distribution of the smallest and second smallest eigenvalues at the hard edge with unitary symmetry. It was derived in [14] that

$$\begin{aligned}
 p_{(2)}^{\text{hard}, a}(s-z, s) &= \frac{z^2 s^a (s-z)^a e^{-s/4}}{4^{2a+3} \Gamma(a+1) \Gamma(a+2) \Gamma^2(a+3)} \\
 &\times \exp\left(\int_0^s \frac{dr}{r} [v(r) + 2C(r)]\right) (u\partial_z v - v\partial_z u).
 \end{aligned} \tag{2.1}$$

Here $v(s)$ is the solution of the second-order, second-degree ODE ($' \equiv d/ds$) – a variant of the σ -form of the third Painlevé equation, [14, equation (5.25)]

$$s^2(v'')^2 - (a+2)^2(v')^2 + v'(4v' - 1)(sv' - v) + \frac{1}{2}a(a+2)v' - \frac{1}{16}a^2 = 0, \tag{2.2}$$

satisfying the boundary conditions [14, equation (5.22)]. Important to our subsequent workings is the fact that $p_{(1)}^{\text{hard}, a}(s)$ —the probability density function for the smallest eigenvalue at the hard edge of an ensemble with unitary symmetry—can be expressed in terms of $v(s)$ by [11, 13, equation (8.93)]

$$p_{(1)}^{\text{hard}, a}(s) = \frac{s^a}{2^{2a+2} \Gamma(a+1) \Gamma(a+2)} \exp\left(\int_0^s \left(v(t) - \frac{t}{4}\right) \frac{dt}{t}\right). \tag{2.3}$$

To define $C(s)$, introduce the auxiliary quantity $\mu = \mu(s)$ according to [14, equation (5.25)],

$$\mu + s = 4sv'. \tag{2.4}$$

Then, according to [14, equation (5.20)], C is specified by

$$2C + a + 3 = s \frac{\mu' - 2}{\mu}. \tag{2.5}$$

These quantities are closely related to the Hamiltonian variables of Okamoto’s theory for PIII’, as will be seen subsequently.

The variables $u(z; s)$ and $v(z; s)$ are the components of a solution to the associated isomonodromic problem for the generic third Painlevé equation or the degenerate fifth Painlevé equation. They satisfy the Lax pair [14, equations (5.34-7)], on that domain $s > z$, $s, z \in \mathbb{R}$, with real $a > -1$, $a \in \mathbb{R}$,

$$z(s - z)\partial_z u = -Czu - (\mu + z)v, \tag{2.6}$$

$$z(s - z)\partial_z v = -z\left[\xi + \frac{1}{4}(z - s)\right]u + [-2s + (C + a + 2)z]v, \tag{2.7}$$

and

$$(s - z)s\partial_s u = zCu + (\mu + s)v, \tag{2.8}$$

$$(s - z)s\partial_s v = z\xi u - [s(2C + a) - zC]v, \tag{2.9}$$

where ξ is a further auxiliary quantity specified by [14, equation (5.19)]

$$\xi = -\frac{sC(C + a)}{\mu + s}. \tag{2.10}$$

For (2.6)–(2.9) to specify a unique solution appropriate boundary conditions must be specified. Their explicit form can be found in [14].

2.2. Okamoto PIII’ theory

We seek to make the links to the Hamiltonian theory of the third Painlevé equation in order to draw upon the results of Okamoto [24, 25] and the work by Forrester and Witte [13]. As given in these works the Hamiltonian theory of Painlevé III’ can be formulated in the variables $\{q, p; s, H\}$ where the Hamiltonian itself is given by ($' \equiv d/ds$)

$$sH = q^2 p^2 - (q^2 + v_1 q - s)p + \frac{1}{2}(v_1 + v_2)q. \tag{2.11}$$

With H so specified the corresponding Hamilton equations of motion are

$$sq' = 2q^2 p - (q^2 + v_1 q - s), \tag{2.12}$$

$$sp' = -2qp^2 + (2q + v_1)p - \frac{1}{2}(v_1 + v_2). \tag{2.13}$$

From these works it is known that the canonical variables can be found from the time evolution of the Hamiltonian itself by

$$p = h' + \frac{1}{2}, \tag{2.14}$$

$$q = \frac{sh'' - v_1 h' + \frac{1}{2}v_2}{\frac{1}{2}(1 - 4(h')^2)}, \tag{2.15}$$

where

$$h = sH + \frac{1}{4}v_1^2 - \frac{1}{2}s. \tag{2.16}$$

In turn the Painlevé III’ σ -function is related to the Hamiltonian by

$$\sigma_{\text{III}'}(s) := -(sH)\Big|_{s \rightarrow s/4} - \frac{1}{4}v_1(v_1 - v_2) + \frac{1}{4}s. \tag{2.17}$$

In the work [14] (see proposition 5.21) the identification made with the Painlevé III’ system gave the parameter correspondence $v_1 = a + 2$, $v_2 = a - 2$ and

$$v(s) = -\sigma_{\text{III}'}(s) + \frac{1}{4}s - a - 2. \tag{2.18}$$

The quantity C appearing in (2.1) and the auxiliary quantities μ and ξ can be related to p and q in the corresponding Hamiltonian system.

Proposition 1. *The variables μ , C , ξ are related to the canonical Painlevé III' co-ordinates by*

$$\frac{\mu}{s} = (p - 1)|_{s \mapsto s/4}, \quad (2.19)$$

$$C = -qp|_{s \mapsto s/4}, \quad (2.20)$$

$$\xi = q(a - qp)|_{s \mapsto s/4}. \quad (2.21)$$

Proof. From equation (5.21) of [14] and (2.18) we compute that

$$v(s) = h(s/4) - \frac{1}{4}(a + 2)^2 + \frac{1}{8}s. \quad (2.22)$$

Differentiating this and employing the relations (2.4) and (2.14) we find (2.19). Using (2.5) we note that $4s^2v'' = 2s + (2C + a + 2)\mu$ and with the above equation and (2.15) we deduce (2.20). Equation (2.21) then follows from (2.10). \square

For the Hamiltonian (2.11), Okamoto [25] has identified two Schlesinger transformations with the property

$$T_1(v_1, v_2) = (v_1 + 1, v_2 + 1), \quad T_2(v_1, v_2) = (v_1 + 1, v_2 - 1), \quad (2.23)$$

and has furthermore specified the corresponding mapping of p and q . Recalling (v_1, v_2) in terms of a above (2.18), we see that in the present case T_1 corresponds to $a \mapsto a + 1$. Reading from [13] equation (4.40-3) gives the following result.

Proposition 2 ([13, equations (4.40-3)]). *The Painlevé III' canonical variables $q[a](s)$, $p[a](s)$ satisfy coupled recurrence relations in a*

$$q[a + 1] = -\frac{s}{q[a]} + \frac{(a + 1)s}{q[a](q[a](p[a] - 1) - 2) + s}, \quad (2.24)$$

$$p[a + 1] = \frac{1}{s}q[a](q[a](p[a] - 1) - 2) + 1. \quad (2.25)$$

The reader should note that we haven't made the scale change $s \mapsto s/4$ here. The initial conditions are given by (2.49) below for the sequence $a \in \mathbb{Z}_{\geq 0}$.

2.3. Isomonodromic system

We now turn our attention to the isomonodromic system (2.6)–(2.9) for u, v associated with the Painlevé system. Following the development of [14] we define the matrix variable

$$\Psi(z; s) = \begin{pmatrix} u(z; s) \\ v(z; s) \end{pmatrix}. \quad (2.26)$$

To begin with our interest is in the recurrence relations that are satisfied by u and v upon the mapping $a \mapsto a + 1$.

Proposition 3. *The isomonodromic components u, v satisfy linear coupled recurrence relations in a*

$$u[a + 1] = \frac{s}{s - z} \left(u[a] - \frac{C[a] + a}{\xi[a]} v[a] \right), \quad (2.27)$$

$$v[a + 1] = -\frac{s}{s - z} \frac{1}{4} \frac{C[a] + a}{\xi[a]} \left(zu[a] - s \frac{C[a] + a}{\xi[a]} v[a] \right). \quad (2.28)$$

The initial conditions are given by (2.50) for the sequence $a \in \mathbb{Z}_{\geq 0}$.

Proof. The result (2.1) from [14] was derived as the hard edge scaling limit of the joint distribution of the first and second eigenvalues in the finite N Laguerre unitary ensemble. To make our derivation self-contained we include some essential definitions and results from [14] concerning the finite n Laguerre unitary ensemble. Consider the *deformed Laguerre weight*

$$w(x; t) := x^2(x + t)^a e^{-x}, \quad x \in [0, \infty), \tag{2.29}$$

and the orthonormal system of polynomials $\{p_n(x; t; a)\}_{n=0}^\infty$ defined by the standard orthogonality relations with respect to the above weight

$$\int_0^\infty dx w(x) p_n(x) x^m = \begin{cases} 0 & 0 \leq m < n \\ h_n & m = n. \end{cases} \tag{2.30}$$

We denote the leading and sub-leading coefficients of $p_n(x; t; a)$ by $\gamma_n, \gamma_{n,1}$, respectively. As with general systems of orthogonal polynomials our system satisfies the three term recurrence relation

$$a_{n+1} p_{n+1}(x) = (x - b_n) p_n(x) - a_n p_{n-1}(x), \quad n \geq 1, \tag{2.31}$$

which serves to define the tridiagonal coefficients a_n, b_n . However it turns out that the latter coefficients are not suitable co-ordinates and we observe that the set

$$\theta_n := 2n + a + 3 - t - b_n, \tag{2.32}$$

$$\kappa_n := (n + 1)t - a_n^2 - \frac{\gamma_{n,1}}{\gamma_n}, \tag{2.33}$$

feature directly in the Painlevé theory. Furthermore we need to work with the orthogonal polynomial ratios

$$Q_n(x; t) := \frac{p_n(x; t; a)}{p_n(0; t; a)}, \tag{2.34}$$

rather than the polynomials themselves along with a partner function

$$R_n := Q_n - Q_{n-1}. \tag{2.35}$$

It is these latter quantities that possess well-defined limits under the hard edge scaling

$$\lim_{n \rightarrow \infty} 4n\theta_n(t)|_{t=s/4n} = \mu(s), \tag{2.36}$$

$$\lim_{n \rightarrow \infty} \kappa_n(t)|_{t=s/4n} = -\frac{1}{4}\mu(s). \tag{2.37}$$

along with

$$\lim_{n \rightarrow \infty} Q_n(x; t)|_{x=-z/4n, t=s/4n} = u(z; s), \tag{2.38}$$

$$\lim_{n \rightarrow \infty} nR_n(x; t)|_{x=-z/4n, t=s/4n} = v(z; s), \tag{2.39}$$

as given by equation (5.10) for θ_N , equation (5.12) for κ_N , equation (5.28) for Q_N and equation (5.29) for R_N in [14].

In the finite N Laguerre unitary ensemble the transformation $a \mapsto a + 1$ implies a Christoffel–Uvarov transformation of the weight $w(x) \mapsto (x + t)w(x)$. From the work of Uvarov [31] we deduce that the orthogonal polynomials $p_N(x; t; a)$ (we adopt the conventions and notations of section 2 in [14], which should not be confused with their subsequent use in section 3) transform

$$\hat{p}_N := p_N(x; t; a + 1) = \frac{A_N^{(1,0)}}{x + t} [p_{N+1}(x; t; a)p_N(-t; t; a) - p_N(x; t; a)p_{N+1}(-t; t; a)], \tag{2.40}$$

where $A_N^{(1,0)}$ is a normalization. In the notations of [14] the three term recurrence coefficients transform as

$$\hat{a}_N^2 = a_N^2 \frac{\gamma_N^2}{\gamma_{N+1}\gamma_{N-1}} \frac{p_{N+1}(-t; t; a)p_{N-1}(-t; t; a)}{p_N(-t; t; a)^2}, \quad (2.41)$$

$$\hat{b}_N = b_N + a_N \frac{p_{N-1}(-t; t; a)}{p_N(-t; t; a)} - a_{N+1} \frac{p_N(-t; t; a)}{p_{N+1}(-t; t; a)}. \quad (2.42)$$

Employing the variables Q_N, R_N (see the definitions equations (3.41) and (3.52) of [14]) instead of p_N, p_{N-1} we find that the transformation gives

$$\hat{Q}_N(x) = \frac{t}{x+t} \left[Q_N(x) - \frac{Q_N(-t)}{R_{N+1}(-t)} R_{N+1}(x) \right], \quad (2.43)$$

$$\hat{R}_N(x) = \frac{t}{x+t} Q_N(-t) \left[\frac{R_N(x)}{R_N(-t)} - \frac{R_{N+1}(x)}{R_{N+1}(-t)} \right]. \quad (2.44)$$

However the second of these equations will suffer a severe cancellation under the hard edge scaling limit $t \rightarrow s/4N, x \rightarrow -z/4N$ as $N \rightarrow \infty$ so we need to be able to handle the subtle cancellations occurring. For this we employ a restatement of the identity equation (3.42) of [14]

$$x\theta_N Q_N + (\kappa_N - t)R_N - (\kappa_{N+1} + t)R_{N+1} = 0, \quad (2.45)$$

which gives us an exact relation between R_N and R_{N+1} . We now compute

$$\hat{R}_N(x) = -\frac{t}{x+t} \frac{\theta_N Q_N(-t)}{R_N(-t)} \frac{t Q_N(-t) R_N(x) + x R_N(-t) Q_N(x)}{(\kappa_N - t)R_N(-t) - t\theta_N Q_N(-t)}. \quad (2.46)$$

We are now in a position to take the hard edge scaling limits (2.36), (2.37), (2.38) and (2.39). In addition we employ the identity, [14, equation (5.45)]

$$\frac{v(s; s)}{u(s; s)} = \frac{\xi}{C+a} = -\frac{sC}{\mu+s}. \quad (2.47)$$

The final result is (2.27) and (2.28) where all dependencies other than a are suppressed. \square

2.4. Special Case $a \in \mathbb{Z}$

In section 5.2 of [14] determinantal evaluations were given of the Painlevé variables v, μ, C and ξ ; of the isomonodromic components u and v ; and of A_a for $a \in \mathbb{Z}_{\geq 0}$. These were of Toeplitz or bordered Toeplitz form and of sizes $a \times a, (a+1) \times (a+1)$ and $(a+2) \times (a+2)$, respectively. Here we content ourselves with displaying the first two cases only, which can serve as initial conditions for the recurrences in propositions 2 and 3. In order to signify the a -value we append a subscript to the variables. In all that follows $I_\sigma(z)$ refers to the standard modified Bessel function with index σ and argument z , see section 10.25 of [26].

2.4.1. $a = 0$. Some details of the first case $a = 0$ were given in propositions 5.9, 5.10 and 5.11 of [14] and we augment that collection by computing the remaining variables. Thus we find for the primary variables

$$\nu_0(s) = 0, \quad \mu_0(s) = -s, \quad C_0(s) = 0, \quad \xi_0(s) = 0, \quad (2.48)$$

for the canonical Hamiltonian variables

$$p_0(s/4) = 0, \quad q_0(s/4) = \frac{\sqrt{s}}{2} \frac{I_3(\sqrt{s})}{I_2(\sqrt{s})}, \quad (2.49)$$

the isomonodromic components

$$u_0(z; s) = \frac{8}{z} I_2(\sqrt{z}), \quad v_0(z; s) = \frac{4}{\sqrt{z}} I_3(\sqrt{z}), \tag{2.50}$$

and the distribution of the spacing

$$A_0(z) = \frac{1}{4} e^{-z/4} [I_2(\sqrt{z})^2 - I_1(\sqrt{z}) I_3(\sqrt{z})]. \tag{2.51}$$

This formula is essentially the same as the gap probability at the hard edge for $a = 2$, as one can see from the $\mu = 0$ specialization of equation (8.97) in [11]. Interestingly we should point out that the moments of the above distribution can be exactly evaluated and we illustrate this observation by giving the first few examples ($m_0 = 1$)

$$\begin{aligned} m_1 &= 4e^2 [I_0(2) - I_1(2)], \\ m_2 &= 32e^2 I_0(2), \\ m_3 &= 384e^2 [2I_0(2) + I_1(2)], \\ m_4 &= 2048e^2 [13I_0(2) + 9I_1(2)], \\ m_5 &= 20480e^2 [55I_0(2) + 42I_1(2)], \\ m_6 &= 98304e^2 [557I_0(2) + 441I_1(2)]. \end{aligned}$$

2.4.2. $a = 1$. This case was not considered in [14]. We have computed these from the results for the finite rank deformed Laguerre ensemble, as given in section 4 of [14], and then applied the hard edge scaling limits given by the Hilb type asymptotic formula equation (5.2) therein and the limits of proposition 5.1 and corollary 5.2 of [14]. For the primary variables we find

$$v_1(s) = \frac{\sqrt{s} I_3(\sqrt{s})}{2 I_2(\sqrt{s})}, \tag{2.52}$$

$$\mu_1(s) = -4\sqrt{s} \frac{I_3(\sqrt{s})}{I_2(\sqrt{s})} - s \frac{I_3(\sqrt{s})^2}{I_2(\sqrt{s})^2}, \tag{2.53}$$

$$C_1(s) = -3 + \frac{\sqrt{s} I_2(\sqrt{s})}{2 I_3(\sqrt{s})} - \frac{\sqrt{s} I_3(\sqrt{s})}{2 I_2(\sqrt{s})}, \tag{2.54}$$

$$\xi_1(s) = \frac{s}{4} - \frac{s I_2(\sqrt{s})^2}{4 I_3(\sqrt{s})^2} + \frac{3\sqrt{s} I_2(\sqrt{s})}{2 I_3(\sqrt{s})}, \tag{2.55}$$

the PIII' canonical Hamiltonian variables

$$p_1(s/4) = 1 - \frac{I_3(\sqrt{s}) I_1(\sqrt{s})}{I_2(\sqrt{s})^2}, \tag{2.56}$$

$$q_1(s/4) = \frac{I_2(\sqrt{s})}{2 I_3(\sqrt{s})} \frac{\sqrt{s} I_2(\sqrt{s})^2 - 6 I_2(\sqrt{s}) I_3(\sqrt{s}) - \sqrt{s} I_3(\sqrt{s})^2}{I_1(\sqrt{s}) I_3(\sqrt{s}) - I_2(\sqrt{s})^2}, \tag{2.57}$$

the isomonodromic components for generic argument $s > z > 0$

$$u_1(z; s) = \frac{8\sqrt{s}}{z I_3(\sqrt{s})} \frac{\sqrt{s} I_1(\sqrt{s}) I_2(\sqrt{z}) - \sqrt{z} I_1(\sqrt{z}) I_2(\sqrt{s})}{s - z}, \tag{2.58}$$

$$v_1(z; s) = \frac{4\sqrt{s} I_2(\sqrt{s})}{\sqrt{z} I_3(\sqrt{s})^2} \frac{\sqrt{s} I_2(\sqrt{s}) I_3(\sqrt{z}) - \sqrt{z} I_2(\sqrt{z}) I_3(\sqrt{s})}{s - z}, \tag{2.59}$$

and the isomonodromic components on $s = z$

$$u_1(s; s) = -\frac{4 I_1(\sqrt{s})}{\sqrt{s}} + \frac{4 I_2(\sqrt{s})^2}{\sqrt{s} I_3(\sqrt{s})}, \tag{2.60}$$

$$v_1(s; s) = -2 I_2(\sqrt{s}) \frac{-s I_1(\sqrt{s})^2 + 2\sqrt{s} I_1(\sqrt{s}) I_2(\sqrt{s}) + (8 + s) I_2(\sqrt{s})^2}{[\sqrt{s} I_1(\sqrt{s}) - 4 I_2(\sqrt{s})]^2}, \tag{2.61}$$

and the distribution of the eigenvalue gap is

$$\begin{aligned}
 A_1(z) = & 2^{-4} [I_0(\sqrt{z})I_2(\sqrt{z}) - I_1(\sqrt{z})^2] \int_z^\infty ds e^{-s/4} I_2(\sqrt{s}) \\
 & + 2^{-3} z^{-1/2} I_2(\sqrt{z}) \int_z^\infty ds \sqrt{s} e^{-s/4} \frac{[\sqrt{s}I_1(\sqrt{z})I_2(\sqrt{s}) - \sqrt{z}I_1(\sqrt{s})I_2(\sqrt{z})]}{s - z}.
 \end{aligned}
 \tag{2.62}$$

From the point of view of checking one can verify that the above solutions satisfy their respective characterizing equations.

2.5. Lax pairs

We now examine the isomonodromic system from the viewpoint of its characterization as the solution to the partial differential systems with respect to z and s .

Proposition 4 ([14, equations (5.51, 5.52, 5.54-7)]). *The matrix form of the spectral derivatives (2.6) and (2.7) and deformation derivatives (2.8) and (2.9) yield the Lax pair*

$$\partial_z \Psi = \left\{ \begin{pmatrix} 0 & 0 \\ \frac{1}{4} & 0 \end{pmatrix} + \begin{pmatrix} C & \frac{\mu + s}{s} \\ \xi & -C - a \end{pmatrix} \frac{1}{z - s} + \begin{pmatrix} 0 & -\frac{\mu}{s} \\ 0 & -2 \end{pmatrix} \frac{1}{z} \right\} \Psi,
 \tag{2.63}$$

and

$$\partial_s \Psi = \left\{ \frac{1}{s} \begin{pmatrix} -C & 0 \\ -\xi & -C \end{pmatrix} - \begin{pmatrix} C & \frac{\mu + s}{s} \\ \xi & -C - a \end{pmatrix} \frac{1}{z - s} \right\} \Psi.
 \tag{2.64}$$

This system is essentially equivalent to the isomonodromic system of the fifth Painlevé equation but is the degenerate case. The system has two regular singularities at $z = 0, s$ and an irregular one at $z = \infty$ with a Poincaré index of $\frac{1}{2}$.

The form of the isomonodromic system (2.6)–(2.10) is not suitable for computing the hard-to-soft edge scaling limit, so we need to perform some preliminary transformations on it.

Proposition 5. *Under the gauge transformation*

$$u, v \mapsto z^{-1} s^{a/2} (s - z)^{-a/2} u, v
 \tag{2.65}$$

the spectral derivatives (2.6) and (2.7) become

$$z(s - z)\partial_z u = [s - z - z(C + \frac{1}{2}a)]u - (\mu + z)v,
 \tag{2.66}$$

$$z(s - z)\partial_z v = -z[\xi + \frac{1}{4}(z - s)]u + [z - s + (C + \frac{1}{2}a)z]v,
 \tag{2.67}$$

whilst the deformation derivatives (2.8) and (2.9) become

$$(s - z)s\partial_s u = (C + \frac{1}{2}a)zu + (\mu + s)v,
 \tag{2.68}$$

$$(s - z)s\partial_s v = z\xi u + (C + \frac{1}{2}a)(z - 2s)v.
 \tag{2.69}$$

Furthermore let us scale the spectral variable $z \rightarrow sr$. Consequently equations (2.66) and (2.67) become

$$\partial_r \Psi = \left\{ \begin{pmatrix} 0 & 0 \\ \frac{1}{4}s & 0 \end{pmatrix} + \begin{pmatrix} C + a/2 & \frac{\mu + s}{s} \\ \xi & -C - a/2 \end{pmatrix} \frac{1}{r - 1} + \begin{pmatrix} 1 & -\frac{\mu}{s} \\ 0 & -1 \end{pmatrix} \frac{1}{r} \right\} \Psi,
 \tag{2.70}$$

and equations (2.68) and (2.69) become

$$s\partial_s \Psi = \left\{ \begin{pmatrix} -C - a/2 & 0 \\ -\xi & -C - a/2 \end{pmatrix} - \begin{pmatrix} C + a/2 & \frac{\mu + s}{s} \\ \xi & -C - a/2 \end{pmatrix} \frac{1}{r - 1} \right\} \Psi.
 \tag{2.71}$$

This system has two regular singularities $r = 0, 1$ and an irregular one at $r = \infty$ with Poincaré rank of $\frac{1}{2}$ (due to the nilpotent character of the leading matrix in (2.70)), and is denoted by the symbol $(1)^2(\frac{3}{2})$.

Remark 1. The symbol $(1)^2(\frac{3}{2})$ encodes data about the local solution of (2.70) in the neighbourhoods of the regular singularities $r = 0, 1$, which have Poincaré ranks of 1, and that of the irregular singularity at $r = \infty$. In the latter case the Poincaré rank is $\frac{3}{2}$, which comes about from the fact that the second order scalar ODE in its *normal form*, or being of *SL type*

$$\frac{d^2}{dr^2}\tilde{u} + Q\tilde{u} = 0, \tag{2.72}$$

where the coefficient $Q \sim -1/(4r)$ as $r \rightarrow \infty$, which means that solutions possess the asymptotic behaviour $\tilde{u} \sim \exp(\pm r^{1/2})$.

The precise solutions we seek are defined by their local expansions about $r = 0, 1$, and in particular the former case. From the general theory of linear ODE [6, 29] we can deduce the existence of convergent expansions about $r = 0$ ($s \neq 0$)

$$u(r; s) = \sum_{m=0}^{\infty} u_m(s)r^{\chi_0+m}, \tag{2.73}$$

$$v(r; s) = \sum_{m=0}^{\infty} v_m(s)r^{\chi_0+m}, \tag{2.74}$$

with a radius of convergence of at most unity. The indicial values χ_0 are fixed by

$$\mu v_0 + su_0(\chi_0 - 1) = 0, \quad v_0(\chi_0 + 1) = 0, \tag{2.75}$$

and the appropriate solution has $v_0 = 0, u_0 \neq 0$ and $\chi_0 = 1$ (actually from [14, equations (5.38), (5.39)] we know $u_0 = s, v_0 = 0$). The general coefficients are given by the recurrence relations

$$4m(m+2)su_m = -[-4(m^2 + m - 2)s + 2(m+2)s(2C+a) + \mu(s - 4\xi)]u_{m-1} + s\mu u_{m-2} - 2[2(m+2)s + (2C+a)\mu + 2(m+1)\mu]v_{m-1}, \tag{2.76}$$

$$4(m+2)v_m = (s - 4\xi)u_{m-1} - su_{m-2} + 2[2C+a + 2(m+1)]v_{m-1}. \tag{2.77}$$

The first few terms are given by

$$u_0 = s, \quad v_0 = 0, \tag{2.78}$$

$$u_1 = -\frac{1}{2}(2C+a)s + \frac{1}{3}\mu(\xi - \frac{1}{4}s), \quad v_1 = -\frac{1}{3}s(\xi - \frac{1}{4}s). \tag{2.79}$$

Similar considerations apply to the local expansions about $r = 1$ however in the hard-to-soft edge limit this singularity will diverge to ∞ and we will not be able to draw any simple conclusions in this case.

3. Hard to soft edge scaling

The hard edge to soft edge scaling limit [4] will be interpreted as the degeneration of PIII' to PII. Therefore we begin with a summary of the relevant Okamoto theory for PII.

3.1. Okamoto PII theory

Henceforth the canonical variables of the Hamiltonian system for PII will be denoted by $\{q, p; t, H\}$ and should not be confused with the use of the same symbols for PIII'. Conforming to common usage we have the parameter relations $\alpha = \alpha_1 - \frac{1}{2} = \frac{1}{2} - \alpha_0$. The PII Hamiltonian is

$$H = -\frac{1}{2}(2q^2 - p + t)p - \alpha_1 q, \quad (3.1)$$

and therefore the PII Hamilton equations of motion ($\dot{\cdot} \equiv d/dt$) are

$$\dot{q} = p - q^2 - \frac{1}{2}t, \quad \dot{p} = 2qp + \alpha_1. \quad (3.2)$$

The transcendent $q(t; \alpha)$ then satisfies the standard form of the second Painlevé equation

$$\ddot{q} = 2q^3 + tq + \alpha. \quad (3.3)$$

The PII Hamiltonian $H(t)$ satisfies the second-order second-degree differential equation of Jimbo–Miwa–Okamoto σ form for PII,

$$(\ddot{H})^2 + 4(\dot{H})^3 + 2\dot{H}[t\dot{H} - H] - \frac{1}{4}\alpha_1^2 = 0. \quad (3.4)$$

Using the first two derivatives of the non-autonomous Hamiltonian H

$$\begin{aligned} \dot{H} &= -\frac{1}{2}p, \\ \ddot{H} &= -qp - \frac{1}{2}\alpha_1, \end{aligned} \quad (3.5)$$

we can recover the canonical variables of (3.2).

3.2. Degeneration from PIII' to PII

We know from [4] that upon the scaling (1.11) of the variables and taking $a \rightarrow \infty$ the hard edge kernel (1.10) limits to the soft edge kernel (1.2) and furthermore the joint distribution $p_{(2)}^{\text{hard}, a}(x_1, x_2)$ limits to $p_{(2)}^{\text{soft}}(x_1, x_2)$. The same holds true for the relationship between $p_{(1)}^{\text{hard}, a}(s)$ and $p_{(1)}^{\text{soft}}(s)$. This latter fact helps in our computation of $p_{(2)}^{\text{soft}}(x_1, x_2)$, since the evaluation of $p_{(1)}^{\text{soft}}(s)$ in terms of PII is known from previous work [11, 12] allowing the limiting form of $v(t)$ in (2.3) to be deduced. But $v(t)$ is the very same PIII' quantity appearing in the evaluation (2.1) of $p_{(2)}^{\text{hard}, a}(s - z, s)$.

Proposition 6. *Let*

$$s = a^2[1 - 2^{2/3}a^{-2/3}\tau]. \quad (3.6)$$

We have that for $a \rightarrow \infty$

$$v(s) - \frac{s}{4} + a \rightarrow -\left(\frac{a}{2}\right)^{2/3} \sigma_{II}(\tau), \quad (3.7)$$

where $t = -2^{1/3}\tau$,

$$\sigma_{II}(\tau) = -2^{1/3} H(t)|_{\alpha_1=2}, \quad (3.8)$$

and furthermore

$$\sigma_{II}(\tau) \underset{\tau \rightarrow \infty}{\sim} \frac{d}{d\tau} \log K^{\text{soft}}(\tau, \tau). \quad (3.9)$$

Proof. We know from [11, equation (8.84)] that

$$p_{(1)}^{\text{soft}}(s) = \rho_{(1)}^{\text{soft}}(s) \exp\left(-\int_s^\infty \left(\sigma_{II}(t) - \frac{d}{dt} \log \rho_{(1)}^{\text{soft}}(t)\right) dt\right), \quad (3.10)$$

where $\rho_{(1)}^{\text{soft}}(s) = K^{\text{soft}}(s, s)$. On the other hand

$$p_{(1)}^{\text{soft}}(s) = \lim_{a \rightarrow \infty} 2^{2/3} a^{4/3} p_{(1)}^{\text{hard}, a}(a^2[1 - 2^{2/3} a^{-2/3} s]). \tag{3.11}$$

Substituting (2.3) in the rhs and (3.10) in the lhs of (3.11), and comparing the respective large s forms implies (3.7). The boundary condition (3.9) is immediate from (3.10). \square

It will be shown in the appendix that the solution of the σ form of PII (3.4) with $\alpha_1 = 2$ as required by (3.8), and subject to the boundary condition (3.9), can be generated from the well known Hastings–McLeod solution of PII.

The scaled form of C in (2.1), as well as the auxiliary quantities μ (2.4) and ξ (2.10) can now be found as a consequence of (3.7).

Proposition 7. *Let s be related to τ by (3.6), and define $t = -2^{1/3}\tau$ as before. As $a \rightarrow \infty$*

$$\mu(s) \rightarrow -2^{1/3} a^{4/3} p(t), \tag{3.12}$$

$$2C(s) + a \rightarrow 2^{2/3} a^{2/3} \left[q(t) + \frac{2}{p(t)} \right], \tag{3.13}$$

$$\xi(s) \rightarrow \frac{1}{4} a^2 - 2^{-2/3} a^{4/3} \left[\left(q(t) + \frac{2}{p(t)} \right)^2 - \frac{1}{2} p(t) \right]. \tag{3.14}$$

Proof. Simple calculations using (3.7) and (2.4), (2.5) and (2.10) give (3.12), (3.13) and (3.14), respectively. \square

Now we turn to task of deducing the appropriate scaling of the associated linear systems and their limits in the hard edge to soft edge transition. There are a handful of references treating the problem of how the degeneration scheme of the Painlevé equations is manifested from the viewpoint of isomonodromic deformations. In comparison to the work [19] our situation is that of the degenerate PV case with nilpotent matrix A_∞ as given by equation (11) in that work and its reduction to the case of equation (13), again with nilpotent matrix A_∞ , which corresponds to PII. In the more complete examination of the coalescence scheme, as given in [22], our reduction is the limit of the degenerate PV (P5-B case) to that P34, and therefore equivalent to PII. However many details we require are missing or incomplete in [19, 22], so we give a fuller account of this scaling and limit for our example.

Lemma 1. *Let $\epsilon = 2^{1/6} a^{-1/3}$. The independent spectral variable scales as $r = \epsilon^2 x$. Under the hard-to-soft edge scaling limit $\epsilon \rightarrow 0$ the isomonodromic components scale as $u = O(\epsilon^{-4})$ and $v = O(\epsilon^{-6})$.*

Proof. Let us denote the leading order scaling of the expansion coefficients given in (2.73) and (2.74) by $u_m = O(a^{\omega_m})$ and $v_m = O(a^{\lambda_m})$. Employing the leading order terms of the auxiliary variables (3.12), (3.13) and (3.14) in the recurrence relations for the coefficients (2.76) and (2.77) we deduce that

$$\begin{aligned} \omega_m &= \max\{\omega_{m-1} + \frac{2}{3}, \omega_{m-2} + \frac{4}{3}, \lambda_{m-1}\}, \\ \lambda_m &= \max\{\omega_{m-1} + \frac{4}{3}, \omega_{m-2} + 2, \lambda_{m-1} + \frac{2}{3}\}. \end{aligned}$$

In fact all terms on the right-hand side balance each other and are satisfied by the single relation $\omega_m = \omega_{m-1} + \frac{2}{3} = \lambda_{m-1}$. The solution to these is $\omega_m = \frac{2}{3}m + 2, \lambda_m = \frac{2}{3}m + \frac{8}{3}$, given the initial condition $\omega_1 = \frac{8}{3}$. We then deduce that each term in the expansions has leading order $u_m r^{m+1} = O(\epsilon^{-4}), v_m r^{m+1} = O(\epsilon^{-6})$, independent of m . Given that the expansions converge uniformly then the whole sums have the stated leading order expansions. \square

Proposition 8. Let the isomonodromic components scale as $u(r; s) = U(x; t)$, $v(r; s) = \epsilon^{-2}V(x; t)$, as only the relative leading orders matter. As $\epsilon \rightarrow 0$ the spectral derivative scales to one of the Lax pair for the second Painlevé equation $t, x \in \mathbb{R}$

$$\partial_x \begin{pmatrix} U \\ V \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & 0 \end{pmatrix} x + \begin{pmatrix} -q - \frac{2}{p} & -1 \\ \frac{1}{2}(t-p) + \left[q + \frac{2}{p}\right]^2 & q + \frac{2}{p} \end{pmatrix} + \begin{pmatrix} 1 & p \\ 0 & -1 \end{pmatrix} \frac{1}{x} \right\} \begin{pmatrix} U \\ V \end{pmatrix}, \quad (3.15)$$

and the deformation derivative scales to

$$\partial_t \begin{pmatrix} U \\ V \end{pmatrix} = \left\{ \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 1 \\ 0 & -2 \left[q + \frac{2}{p} \right] \end{pmatrix} \right\} \begin{pmatrix} U \\ V \end{pmatrix}. \quad (3.16)$$

Proof. Our starting point is the Lax pair for the degenerate fifth Painlevé system given in Equations (2.70) and (2.71). Using the expansions (3.13), (3.12), (3.14) and (3.6) we deduce that the matrix elements appearing in this pair scale as

$$-C - a/2 + \frac{C+a/2}{r-1} \sim -\epsilon^{-2}h_0, \quad (3.17)$$

$$-C - a/2 - \frac{C+a/2}{r-1} \sim \frac{1}{2}h_0x, \quad (3.18)$$

$$\frac{1}{r} + \frac{C+a/2}{r-1} \sim \epsilon^{-2} \left(\frac{1}{x} - \frac{1}{2}h_0 \right) - \frac{1}{2}(xh_0 + h_1), \quad (3.19)$$

$$\frac{1}{4}s - \frac{s}{\mu+s} \left[(C+a/2)^2 - \frac{1}{4}a^2 \right] \frac{1}{r-1} \sim \epsilon^{-4} \left[\frac{1}{2}(t-x) + \frac{1}{4}h_0^2 - \frac{1}{2}p \right], \quad (3.20)$$

$$-\frac{\mu+s}{s} \frac{1}{r-1} \sim 1, \quad (3.21)$$

$$-\frac{\mu}{sr} + \frac{\mu+s}{s} \frac{1}{r-1} \sim \frac{p}{x} - 1 + \epsilon^2(p-x), \quad (3.22)$$

where the abbreviation is

$$h_0 = 2 \left(q + \frac{2}{p} \right). \quad (3.23)$$

Using the scaling for u, v we deduce a meaningful limit as $\epsilon \rightarrow 0$ given in equations (3.15) and (3.16). One can check that the compatibility of these two equations is ensured by the requirement that q, p satisfy the Hamiltonian equations of motion (3.2). \square

Remark 2. For general α_1 the Lax pair of the PII system is

$$\partial_x Y = \left\{ \begin{pmatrix} 0 & 0 \\ -\frac{1}{2} & 0 \end{pmatrix} x + \begin{pmatrix} -q - \frac{\alpha_1}{p} & -1 \\ \frac{1}{2}(t-p) + \left[q + \frac{\alpha_1}{p} \right]^2 & q + \frac{\alpha_1}{p} \end{pmatrix} + \begin{pmatrix} \frac{1}{2}\alpha_1 & p \\ 0 & -\frac{1}{2}\alpha_1 \end{pmatrix} \frac{1}{x} \right\} Y, \quad (3.24)$$

and

$$\partial_t Y = \left\{ \begin{pmatrix} 0 & 0 \\ \frac{1}{2} & 0 \end{pmatrix} x + \begin{pmatrix} 0 & 1 \\ 0 & -2 \left[q + \frac{\alpha_1}{p} \right] \end{pmatrix} \right\} Y. \quad (3.25)$$

Equation (3.24) has the same form as the nilpotent case of equation (13) of [19], and in addition both members of the Lax pair (3.24) and (3.25) are a variant of the system equation (31), given

subsequently in [19]. It has also been shown in [20] that the Lax pair of this latter system is related to that of Flashka and Newell [8] via an ‘unfolding’ of the spectral variable supplemented by a gauge transformation (see also 5.0.54,5 on page 175 of [9]). In contrast the Lax pair of Jimbo *et al* [18] is not equivalent to any of those mentioned above. In the hard-to-soft edge scaling the regular singularity at $r = 0$ has transformed into the regular singularity at $x = 0$; the regular singularity at $r = 1$ has merged with the irregular one at $r = \infty$ yielding an irregular singularity at $x = \infty$ with its Poincaré rank increased by unity, now being $\frac{3}{2}$. Thus the symbol of the new system is $(1)(\frac{3}{2})$. At the irregular singularity the coefficient of the *normal form* has the behaviour $Q \sim -x/2$ as $x \rightarrow \infty$, which means that solutions possess the asymptotic behaviour $\tilde{U} \sim \exp(\pm \frac{\sqrt{2}}{3}x^{3/2})$.

The solution we seek can be characterized in a precise way though its expansion about the regular singularity $x = 0$.

Lemma 2. *Let us assume $|p(t)| > \delta > 0$ and $t, q(t), p(t)$ lie in compact subsets of \mathbb{C} . The isomonodromic components U, V have a convergent expansion about $x = 0$, with indicial exponent $\chi_0 = 1$, whose leading terms are*

$$\begin{aligned}
 U(x; t) = & 2x + \frac{p^2(2q^2 - p + t) + 2qp - 4}{3p}x^2 \\
 & + \frac{p(p^2(2q^2 - p + t) + 4pq - 8)(2q^2 - p + t) + 2(p^2 - 16q - 4pt)}{48p}x^3 \\
 & + O(x^4),
 \end{aligned} \tag{3.26}$$

$$\begin{aligned}
 V(x; t) = & \frac{p^2(2q^2 - p + t) + 8pq + 8}{3p^2}x^2 \\
 & + \frac{p(p^2(2q^2 - p + t) + 12pq + 24)(2q^2 - p + t) + 2(5p^2 + 16q - 8pt)}{24p^2}x^3 \\
 & + O(x^4).
 \end{aligned} \tag{3.27}$$

Proof. The scaling relations (3.6), (3.7), (3.12), (3.13) and (3.14) can be applied term-wise to the expansions (2.73) and (2.74) along with the explicit results for the leading coefficients (2.78), (2.79). Alternatively one can compute the recurrence relations for the local expansion of the system (3.15)

$$U, V = \sum_{m=0}^{\infty} U_m, V_m x^{\chi_0+m}. \tag{3.28}$$

In such an analysis one finds for the leading relation $p((\chi_0 - 1)U_0 - pV_0) = 0$ and $2p^2(\chi_0 + 1)V_0 = 0$. Clearly for a well-defined solution at $x = 0$ we must have $V_0 = 0$ and so $\chi_0 = 1$ with $U_0 \neq 0$. The recurrence relations for the coefficients are given by

$$\begin{aligned}
 2m(m + 2)pU_m = & [p^2(2q^2 - p + t) + (4 - 2m)qp - 4m]U_{m-1} \\
 & + 2p(pq - m)V_{m-1} - p^2U_{m-2},
 \end{aligned} \tag{3.29}$$

$$\begin{aligned}
 2(m + 2)p^2V_m = & [p^2(2q^2 - p + t) + 8qp + 8]U_{m-1} \\
 & + 2p(pq + 2)V_{m-1} - p^2U_{m-2}.
 \end{aligned} \tag{3.30}$$

Given $U_0 = 2$ these recurrences generate the unique solution stated above. From these recurrence relations it is easy to establish that the local expansions (3.28) define an entire function of x . □

We now have all the preliminary results to obtain the sought Painlevé II evaluation of $p_{(2)}^{\text{soft}}$, specified originally as the Fredholm minor (1.7).

Proposition 9. *Let $p_{(1)}^{\text{soft}}(t)$ be given by (3.10). For some constant C_0 still to be determined, and boundary conditions on U and V still to be determined*

$$\begin{aligned}
 p_{(2)}^{\text{soft}}(t, t-x) &= C_0 p_{(1)}^{\text{soft}}(t) t^{-5/2} \\
 &\times \exp\left(-\frac{4}{3}t^{3/2}\right) \exp\left(\int_{2^{1/3}t}^{\infty} dy \left\{ \left(2q + \frac{4}{p}\right)(-y) - \sqrt{2y} - \frac{5}{2y} \right\}\right) \\
 &\times (U\partial_x V - V\partial_x U)(-2^{1/3}x; -2^{1/3}t).
 \end{aligned}
 \tag{3.31}$$

Proof. Applying the gauge transformation (2.65) to (2.1) and absorbing the pre-factors $\exp(-s/4)$ and s^a into the integral we have

$$\begin{aligned}
 p_{(2)}^{\text{hard},a}(s-z, s) &= \hat{C}_a(s_0) \exp\left(\int_{s_0}^s \frac{dw}{w} \left[v(w) - \frac{w}{4} + a + 2C(w) + a \right]\right) \\
 &\times (u\partial_z v - v\partial_z u)(z; s),
 \end{aligned}$$

where $\hat{C}_a(s_0)$ is a normalization independent of s, z but dependent on a and the reference point s_0 . We are now in a position to apply the limiting forms (3.7) and (3.13) to this, thus obtaining

$$p_{(2)}^{\text{soft}}(t, t-x) = \tilde{C}_0 \exp\left(\int_{2^{1/3}t}^{t_0} dy \left(H + 2q + \frac{4}{p}\right)(-y)\right) (U\partial_x V - V\partial_x U)(-2^{1/3}x; -2^{1/3}t).
 \tag{3.32}$$

To proceed further, we make use of (3.8) and (3.10) to note that for suitable \tilde{C}_0 ,

$$\lim_{t_0 \rightarrow \infty} \tilde{C}_0 \exp\left(\int_{2^{1/3}t}^{t_0} dy H(-y)\right) = p_{(1)}^{\text{soft}}(t).
 \tag{3.33}$$

Furthermore, since

$$p_{(1)}^{\text{soft}}(s) \underset{s \rightarrow \infty}{\sim} \rho_{(1)}^{\text{soft}}(s),
 \tag{3.34}$$

and

$$\rho_{(1)}^{\text{soft}}(s) = K^{\text{soft}}(s, s) \underset{s \rightarrow \infty}{\sim} \frac{1}{8\pi s} \exp\left(-\frac{4}{3}s^{3/2}\right),
 \tag{3.35}$$

we must have

$$H(-y) \underset{y \rightarrow \infty}{\sim} \sqrt{2y} + \frac{1}{y}.
 \tag{3.36}$$

The Hamilton equations (3.5) then imply

$$p(-y) \underset{y \rightarrow \infty}{\sim} \sqrt{\frac{2}{y}} - \frac{2}{y^2},
 \tag{3.37}$$

$$q(-y) \underset{y \rightarrow \infty}{\sim} -\sqrt{\frac{y}{2}} - \frac{3}{4y},
 \tag{3.38}$$

and thus

$$\left(2q + \frac{4}{p}\right)(-y) \underset{y \rightarrow \infty}{\sim} \sqrt{2y} + \frac{5}{2y}.$$

Consequentially, for suitable \tilde{C}_0 ,

$$\begin{aligned} \lim_{t_0 \rightarrow \infty} \tilde{C}_0 \exp \left(\int_{2^{1/3}t}^{t_0} dy \left(2q + \frac{4}{p} \right) (-y) \right) &= t^{-5/2} \exp \left(-\frac{4}{3} t^{3/2} \right) \\ &\times \exp \left(\int_{2^{1/3}t}^{\infty} dy \left\{ \left(2q + \frac{4}{p} \right) (-y) - \sqrt{2y} - \frac{5}{2y} \right\} \right). \end{aligned} \tag{3.39}$$

Substituting (3.33) and (3.39) in (3.32) gives (3.31). □

It remains to specify C_0 in (3.31), and furthermore to specify the $x, t \rightarrow \infty$ asymptotic form of U and V . For this we require the fact, which follows from (1.7), that for $t, x, t - x \rightarrow \infty$,

$$p_{(2)}^{\text{soft}}(t, t - x) \sim \rho_{(1)}^{\text{soft}}(t) \rho_{(1)}^{\text{soft}}(t - x). \tag{3.40}$$

Proposition 10. *In (3.31)*

$$C_0 = \frac{1}{4\pi}, \tag{3.41}$$

and furthermore, for $x, t, t - x \rightarrow \infty$ we have

$$U(-x; -t) \sim a(x, t) \exp \left(\frac{\sqrt{2}}{3} [t^{3/2} - (t - x)^{3/2}] \right), \tag{3.42}$$

$$V(-x; -t) \sim b(x, t) \exp \left(\frac{\sqrt{2}}{3} [t^{3/2} - (t - x)^{3/2}] \right), \tag{3.43}$$

with

$$a(x, t) = \frac{t^{5/4}}{(t - x)^{1/4}}, \tag{3.44}$$

$$b(x, t) = - \left(\sqrt{\frac{t}{2}} - \sqrt{\frac{t - x}{2}} \right) a(x, t). \tag{3.45}$$

Proof. Substituting (3.40) in the lhs of (3.31) and making use of (3.34) and (3.35), it follows from (3.31) that in the asymptotic region in question

$$\begin{aligned} \frac{1}{8\pi(t - x)} \exp \left(-\frac{4}{3} (t - x)^{3/2} \right) \\ \sim C_0 t^{-5/2} \exp \left(-\frac{4}{3} t^{3/2} \right) (U \partial_x V - V \partial_x U) (-2^{1/3}x; -2^{1/3}t). \end{aligned} \tag{3.46}$$

We see immediately from this that (3.42) and (3.43) are valid, for $a(x, t), b(x, t)$ algebraic functions in x and t satisfying

$$\begin{aligned} \frac{1}{8\pi} \frac{t^{5/2}}{t - x} \sim C_0 \left[a(2^{1/3}x, 2^{1/3}t) \partial_y b(-y, 2^{1/3}t) \Big|_{y=-2^{1/3}x} \right. \\ \left. - b(2^{1/3}x, 2^{1/3}t) \partial_y a(-y, 2^{1/3}t) \Big|_{y=-2^{1/3}x} \right]. \end{aligned} \tag{3.47}$$

On the other hand, we read off from (3.15) that for $x, t \rightarrow \infty$

$$\partial_y U(y; -t) \Big|_{y=-x} \sim \left(-q(-t) - \frac{2}{p(-t)} \right) U(-x; -t) - V(-x; -t).$$

Making use of the leading terms in (3.37) and (3.38) as well as (3.42) and (3.43) it follows that (3.45) holds true. This latter formula substituted in (3.47) implies, upon choosing C_0

according to (3.41), that

$$a(x, t)^2 = \frac{t^{5/2}}{\sqrt{t-x}},$$

and (3.44) follows upon taking the positive square root.

We can offer a refinement on the above argument by examining in more detail the isomonodromic system in the asymptotic regime $t \rightarrow -\infty$. In this regime the leading order of the differential equations (3.15) and (3.16) become

$$\partial_x \begin{pmatrix} U \\ V \end{pmatrix} \sim \begin{pmatrix} -\sqrt{-\frac{t}{2}} & -1 \\ -\frac{1}{2}x & \sqrt{-\frac{t}{2}} \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}, \quad (3.48)$$

and

$$\partial_t \begin{pmatrix} U \\ V \end{pmatrix} \sim \begin{pmatrix} 0 & 1 \\ \frac{1}{2}x & -2\sqrt{-\frac{t}{2}} \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}. \quad (3.49)$$

In retaining only the leading order terms we have also implicitly assumed that $x \rightarrow \infty$ because in the regime as $t \rightarrow -\infty$ our solution p is vanishing, which is the location of the only non-zero, finite singularity. Clearly

$$(\partial_x + \partial_t) \begin{pmatrix} U \\ V \end{pmatrix} \sim \begin{pmatrix} -\sqrt{-\frac{t}{2}} & 0 \\ 0 & -\sqrt{-\frac{t}{2}} \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}, \quad (3.50)$$

so that $U(x; t), V(x; t) \sim f(t)g_{1,2}(t-x)$. This means that $\partial_t f = -\sqrt{-\frac{t}{2}}f$ with a solution proportional to $\exp \frac{\sqrt{2}}{3}(-t)^{3/2}$. For the remaining factor we have

$$\begin{pmatrix} g'_1 \\ g'_2 \end{pmatrix} \sim \begin{pmatrix} \sqrt{-\frac{t}{2}} & 1 \\ \frac{1}{2}x & -\sqrt{-\frac{t}{2}} \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad (3.51)$$

or, in terms of the components

$$g''_1 = \frac{1}{2}(x-t)g_1, \quad g_2 = g'_1 - \sqrt{-\frac{t}{2}}g_1. \quad (3.52)$$

Thus, with Ai and Bi the two linearly independent solutions of the Airy equation [26, section 9.2], we have

$$U(x; t) \sim \exp \frac{\sqrt{2}}{3}(-t)^{3/2} \{ \alpha(t)\text{Ai}(2^{-1/3}(x-t)) + \beta(t)\text{Bi}(2^{-1/3}(x-t)) \}, \quad (3.53)$$

and

$$V(x; t) \sim \exp \frac{\sqrt{2}}{3}(-t)^{3/2} \left\{ -\alpha(t) \left[2^{-1/3}\text{Ai}'(2^{-1/3}(x-t)) + \sqrt{-\frac{t}{2}}\text{Ai}(2^{-1/3}(x-t)) \right] \right. \\ \left. - \beta(t) \left[2^{-1/3}\text{Bi}'(2^{-1/3}(x-t)) + \sqrt{-\frac{t}{2}}\text{Bi}(2^{-1/3}(x-t)) \right] \right\}. \quad (3.54)$$

From these we compute

$$U \partial_x V - V \partial_x U \sim \exp 2 \frac{\sqrt{2}}{3}(-t)^{3/2} \left\{ -\frac{1}{2}(x-t) [\alpha\text{Ai} + \beta\text{Bi}]^2 + 2^{-2/3} [\alpha\text{Ai}' + \beta\text{Bi}']^2 \right\}. \quad (3.55)$$

Clearly $\beta = 0$ in order to suppress the dominant terms, and by employing the exponential asymptotics of the Airy function through to second order (the leading order cancels exactly) we have

$$(U\partial_x V - V\partial_x U)(x; t) \sim \frac{2^{-4/3}\alpha^2}{4\pi(x-t)} \exp\left[2\frac{\sqrt{2}}{3}(-t)^{3/2} - 2\frac{\sqrt{2}}{3}(x-t)^{3/2}\right]. \tag{3.56}$$

Employing this into the factorization of $p_{(2)}^{\text{soft}}(t, t-x)$ as $t, t-x, x \rightarrow \infty$ we deduce

$$C_0\alpha(-2^{1/3}t)^2 = 2^{2/3}t^{5/2}, \tag{3.57}$$

and with $C_0 = 1/4\pi$ we infer $\alpha(t) = 2^{11/12}\pi^{1/2}(-t)^{5/4}$. This then precisely reproduces the boundary conditions given by (3.42) and (3.43) along with (3.44) and (3.45). A by-product of this argument is that the isomonodromic components can be represented in the forms

$$U(x; t) \sim -t \frac{\text{Ai}(2^{-1/3}(x-t))}{\text{Ai}(-2^{-1/3}t)}, \tag{3.58}$$

$$V(x; t) \sim t \frac{2^{-1/3}\text{Ai}'(2^{-1/3}(x-t)) + \sqrt{-\frac{t}{2}}\text{Ai}(2^{-1/3}(x-t))}{\text{Ai}(-2^{-1/3}t)}, \tag{3.59}$$

although this is still only valid in the regime $t, t-x, x \rightarrow -\infty$. □

We remark that as written (3.31) is not well defined for $t \leq 0$. In this region we should use instead (3.32), with a suitable t_0 to make use of (3.31) for $t > 0$ so that \tilde{C}_0 can be specified.

In [14] the analogue of (3.31), equation (2.1), was used to provide high precision numerics for the spacing between the two smallest eigenvalues at the hard edge. But to use (3.31) to compute the spacing distribution (1.8) presents additional challenges to obtain control on the accuracy. The essential problem faced in comparison to [14], and in also in comparison to the study [27] relating to partial differential equations (PDEs) based on the Hastings–McLeod PII transcendent, is that our boundary conditions involve algebraic terms, and thus cannot be determined to arbitrary accuracy. In relation to $p(-y)$ and $q(-y)$ this can perhaps be overcome by using the theory of the appendix to map to the Hastings–McLeod solution. But even so, the problem of extending the accuracy of the algebraic terms $a(x, t)$ and $b(x, t)$ in (3.42) and (3.43) would remain. While these points remain under investigation, the problem of determining numerics for (1.8) can be tackled by using a variant of the Fredholm type expansion (1.7), as we will now proceed to detail.

4. Numerical evaluation of moments

We provide some numerical data for the distribution of the spacing of the two largest eigenvalues based on the accurate numerical evaluation of operator determinants that is surveyed in [2]. The joint probability distribution of the two largest eigenvalues is amenable to this method since it is given in terms of a 2×2 operator matrix determinant:

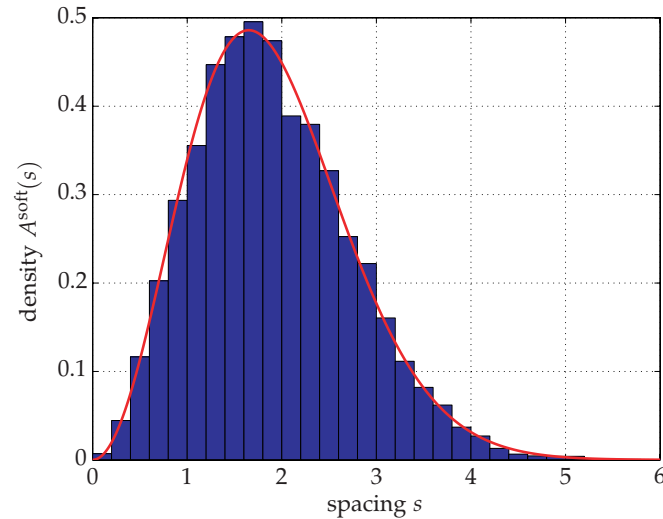
$$F(x, y) = \int_{-\infty}^x \int_{-\infty}^y p_{(2)}^{\text{soft}}(\xi, \eta) d\xi d\eta$$

$$= \begin{cases} E_2^{\text{soft}}(0; (x, \infty)) & (x \leq y), \\ E_2^{\text{soft}}(0; (y, \infty)) - \frac{\partial}{\partial z} \det \left(I - \begin{pmatrix} z K^{\text{soft}} & K^{\text{soft}} \\ z K^{\text{soft}} & K^{\text{soft}} \end{pmatrix} \Big|_{L^2(y,x) \oplus L^2(x,\infty)} \right) \Big|_{z=1} & (x > y). \end{cases}$$

Here, the differentiation with respect to z can be accurately computed by the Cauchy integral formula in the complex domain [3]. It has been used in [2] to calculate the correlation between

Table 1. The first four statistical moments of the density function $A^{\text{soft}}(s)$.

Mean	Variance	Skewness	Excess kurtosis
1.904 350 49	0.683 252 06	0.562 292	0.270 09

**Figure 1.** A plot of the density function $A^{\text{soft}}(s)$ as compared to a histogram of 10 000 draws from a 1000×1000 GUE at the soft edge.

the two largest eigenvalues to 11 digits accuracy

$$\rho \approx 0.505\,647\,231\,59.$$

This number can be calculated without any differentiation of the joint distribution function F since, according to a lemma of Hoeffding [16], the covariance is given by

$$\text{cov} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (F(x, y) - F(x, \infty)F(\infty, y)) \, dx \, dy.$$

In contrast, the spacing distribution

$$G(s) = \int_0^s A^{\text{soft}}(\sigma) \, d\sigma = \int_{-\infty}^{\infty} \frac{\partial}{\partial y} F(x, y)|_{x=y+s} \, dy$$

requires numerical differentiation in the real domain which causes a loss of a couple of digits. The differentiation is done by spectral collocation in Chebyshev points of the first kind and G is numerically represented by polynomial interpolation in the same type of points; table 2 tabulates the values of $A^{\text{soft}}(s)$ and $G(s)$ for $s = 0(0.05)8.80$ to an absolute accuracy of 8 digits based on a polynomial representation of degree 64 that is accurate to about 9 to 10 digits. Figure 1 plots the density function as compared to a histogram obtained from 10 000 draws from a 1000×1000 GUE at the soft edge.

The moments of the random variable S representing the spacing are obtained from the following derivative-free formulae obtained from partial integration:

$$\mathbb{E}(S^n) = \int_0^{\infty} s^n \, dG(s) = n \int_0^{\infty} s^{n-1} (1 - G(s)) \, ds \quad (n = 1, 2, \dots).$$

This way we have obtained the first four statistical moments shown in table 1; estimates of the approximation errors by calculations to higher accuracy indicate the given digits to be correctly

Table 2. Values of the probability density and distribution function for the spacing s between the two largest eigenvalues; with $s = 0(0.05)8.80$.

0.00	0.000 000 00	0.000 000 00	2.95	0.189 221 58	0.888 182 69	5.90	0.000 221 66	0.999 931 69
0.05	0.001 248 77	0.000 020 82	3.00	0.176 894 53	0.897 333 63	5.95	0.000 189 67	0.999 941 95
0.10	0.004 980 37	0.000 166 27	3.05	0.165 047 34	0.905 880 14	6.00	0.000 162 10	0.999 950 73
0.15	0.011 150 87	0.000 559 52	3.10	0.153 696 62	0.913 846 64	6.05	0.000 138 37	0.999 958 23
0.20	0.019 687 90	0.001 320 82	3.15	0.142 854 08	0.921 258 27	6.10	0.000 117 98	0.999 964 62
0.25	0.030 491 79	0.002 566 11	3.20	0.132 526 88	0.928 140 64	6.15	0.000 100 47	0.999 970 07
0.30	0.043 437 13	0.004 405 70	3.25	0.122 718 04	0.934 519 60	6.20	0.000 085 46	0.999 974 71
0.35	0.058 374 84	0.006 943 04	3.30	0.113 426 81	0.940 421 07	6.25	0.000 072 60	0.999 978 65
0.40	0.075 134 43	0.010 273 56	3.35	0.104 649 06	0.945 870 84	6.30	0.000 061 61	0.999 982 00
0.45	0.093 526 60	0.014 483 70	3.40	0.096 377 65	0.950 894 42	6.35	0.000 052 22	0.999 984 84
0.50	0.113 346 13	0.019 650 02	3.45	0.088 602 81	0.955 516 89	6.40	0.000 044 21	0.999 987 25
0.55	0.134 374 84	0.025 838 47	3.50	0.081 312 51	0.959 762 78	6.45	0.000 037 39	0.999 989 28
0.60	0.156 384 76	0.033 103 86	3.55	0.074 492 79	0.963 655 98	6.50	0.000 031 58	0.999 991 00
0.65	0.179 141 31	0.041 489 39	3.60	0.068 128 06	0.967 219 64	6.55	0.000 026 65	0.999 992 46
0.70	0.202 406 46	0.051 026 47	3.65	0.062 201 45	0.970 476 09	6.60	0.000 022 46	0.999 993 68
0.75	0.225 941 92	0.061 734 54	3.70	0.056 695 06	0.973 446 79	6.65	0.000 018 90	0.999 994 71
0.80	0.249 512 15	0.073 621 23	3.75	0.051 590 22	0.976 152 29	6.70	0.000 015 90	0.999 995 58
0.85	0.272 887 21	0.086 682 50	3.80	0.046 867 76	0.978 612 18	6.75	0.000 013 35	0.999 996 31
0.90	0.295 845 51	0.100 903 00	3.85	0.042 508 18	0.980 845 11	6.80	0.000 011 20	0.999 996 92
0.95	0.318 176 24	0.116 256 58	3.90	0.038 491 87	0.982 868 72	6.85	0.000 009 39	0.999 997 43
1.00	0.339 681 57	0.132 706 86	3.95	0.034 799 28	0.984 699 69	6.90	0.000 007 86	0.999 997 86
1.05	0.360 178 60	0.150 207 93	4.00	0.031 411 05	0.986 353 72	6.95	0.000 006 57	0.999 998 22
1.10	0.379 500 94	0.168 705 13	4.05	0.028 308 17	0.987 845 55	7.00	0.000 005 49	0.999 998 53
1.15	0.397 500 12	0.188 135 96	4.10	0.025 472 09	0.989 188 99	7.05	0.000 004 58	0.999 998 78
1.20	0.414 046 48	0.208 430 92	4.15	0.022 884 75	0.990 396 91	7.10	0.000 003 82	0.999 998 99
1.25	0.429 029 96	0.229 514 55	4.20	0.020 528 76	0.991 481 32	7.15	0.000 003 18	0.999 999 16
1.30	0.442 360 44	0.251 306 36	4.25	0.018 387 37	0.992 453 36	7.20	0.000 002 64	0.999 999 31
1.35	0.453 967 85	0.273 721 86	4.30	0.016 444 56	0.993 323 36	7.25	0.000 002 20	0.999 999 43
1.40	0.463 801 99	0.296 673 58	4.35	0.014 685 07	0.994 100 87	7.30	0.000 001 82	0.999 999 53
1.45	0.471 832 10	0.320 071 98	4.40	0.013 094 41	0.994 794 68	7.35	0.000 001 51	0.999 999 61
1.50	0.478 046 18	0.343 826 51	4.45	0.011 658 90	0.995 412 90	7.40	0.000 001 25	0.999 999 68
1.55	0.482 450 10	0.367 846 42	4.50	0.010 365 61	0.995 962 94	7.45	0.000 001 04	0.999 999 73
1.60	0.485 066 56	0.392 041 72	4.55	0.009 202 46	0.996 451 63	7.50	0.000 000 86	0.999 999 78
1.65	0.485 933 85	0.416 323 92	4.60	0.008 158 09	0.996 885 17	7.55	0.000 000 71	0.999 999 82
1.70	0.485 104 54	0.440 606 82	4.65	0.007 221 93	0.997 269 24	7.60	0.000 000 58	0.999 999 85
1.75	0.482 644 01	0.464 807 18	4.70	0.006 384 15	0.997 609 00	7.65	0.000 000 48	0.999 999 88
1.80	0.478 628 94	0.488 845 31	4.75	0.005 635 63	0.997 909 14	7.70	0.000 000 39	0.999 999 90
1.85	0.473 145 81	0.512 645 60	4.80	0.004 967 93	0.998 173 91	7.75	0.000 000 32	0.999 999 92
1.90	0.466 289 29	0.536 136 99	4.85	0.004 373 27	0.998 407 15	7.80	0.000 000 27	0.999 999 93
1.95	0.458 160 68	0.559 253 33	4.90	0.003 844 50	0.998 612 34	7.85	0.000 000 22	0.999 999 95
2.00	0.448 866 44	0.581 933 63	4.95	0.003 375 04	0.998 792 59	7.90	0.000 000 18	0.999 999 96
2.05	0.438 516 64	0.604 122 38	5.00	0.002 958 89	0.998 950 73	7.95	0.000 000 15	0.999 999 96
2.10	0.427 223 62	0.625 769 58	5.05	0.002 590 55	0.999 089 28	8.00	0.000 000 12	0.999 999 97
2.15	0.415 100 61	0.646 830 91	5.10	0.002 265 03	0.999 210 50	8.05	0.000 000 10	0.999 999 98
2.20	0.402 260 50	0.667 267 69	5.15	0.001 977 78	0.999 316 42	8.10	0.000 000 08	0.999 999 98
2.25	0.388 814 72	0.687 046 86	5.20	0.001 724 67	0.999 408 84	8.15	0.000 000 07	0.999 999 98
2.30	0.374 872 23	0.706 140 88	5.25	0.001 501 98	0.999 489 39	8.20	0.000 000 05	0.999 999 99
2.35	0.360 538 55	0.724 527 57	5.30	0.001 306 33	0.999 559 49	8.25	0.000 000 04	0.999 999 99
2.40	0.345 915 04	0.742 189 91	5.35	0.001 134 70	0.999 620 42	8.30	0.000 000 04	0.999 999 99
2.45	0.331 098 15	0.759 115 85	5.40	0.000 984 34	0.999 673 32	8.35	0.000 000 03	0.999 999 99
2.50	0.316 178 92	0.775 298 02	5.45	0.000 852 81	0.999 719 17	8.40	0.000 000 02	0.999 999 99
2.55	0.301 242 47	0.790 733 46	5.50	0.000 737 92	0.999 758 87	8.45	0.000 000 02	1.000 000 00
2.60	0.286 367 70	0.805 423 30	5.55	0.000 637 69	0.999 793 21	8.50	0.000 000 02	1.000 000 00
2.65	0.271 627 06	0.819 372 47	5.60	0.000 550 39	0.999 822 86	8.55	0.000 000 01	1.000 000 00
2.70	0.257 086 34	0.832 589 34	5.65	0.000 474 44	0.999 848 43	8.60	0.000 000 01	1.000 000 00
2.75	0.242 804 70	0.845 085 43	5.70	0.000 408 46	0.999 870 47	8.65	0.000 000 01	1.000 000 00
2.80	0.228 834 62	0.856 875 01	5.75	0.000 351 23	0.999 889 43	8.70	0.000 000 01	1.000 000 00
2.85	0.215 222 06	0.867 974 85	5.80	0.000 301 64	0.999 905 72	8.75	0.000 000 01	1.000 000 00
2.90	0.202 006 58	0.878 403 84	5.85	0.000 258 73	0.999 919 70	8.80	0.000 000 00	1.000 000 00

truncated. The total computing time was 5 h for the solution and 30 h for the higher accuracy control calculation.

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Appendix

As a technical matter we will need to make use of the Gambier or Folding transformation for PII. The fundamental domain or Weyl chamber for the PII system can be taken as the interval $\alpha \in (-\frac{1}{2}, 0]$ or $\alpha \in [0, \frac{1}{2})$, and there exist identities relating the transcendents and related quantities at the endpoints of these intervals. In particular, denoting the transcendent $q(t; \alpha)$ and with $\epsilon^2 = 1$, $t = -2^{1/3}s$ we have [15]

$$\begin{aligned} -\epsilon 2^{1/3} q^2(s; 0) &= \frac{d}{dt} q(t; \tfrac{1}{2}\epsilon) - \epsilon q^2(t; \tfrac{1}{2}\epsilon) - \tfrac{1}{2}\epsilon t, \\ q(t; \tfrac{1}{2}\epsilon) &= \epsilon 2^{-1/3} \frac{1}{q(s; 0)} \frac{d}{ds} q(s; 0). \end{aligned} \quad (\text{A.1})$$

In addition we will employ the Bäcklund transformation theory of PII as formulated by Noumi (see [21]) and put to use in the random matrix context by [12]. We define a shift operator corresponding to a translation of the fundamental weights of the affine Weyl group $A_1^{(1)}$,

$$T_2 : \alpha_0 \mapsto \alpha_0 - 1, \quad \alpha_1 \mapsto \alpha_1 + 1. \quad (\text{A.2})$$

The discrete dynamical system generated by the Bäcklund transformations is also integrable and can be identified with a discrete Painlevé system, discrete dPI. The members of the sequence $\{q[n]\}_{n=0}^\infty$, generated by the shift operator T_2 with the parameters $(\alpha_0 - n, \alpha_1 + n)$, are related by a second-order difference equation which is the alternate form of the first discrete Painlevé equation, a-dPI,

$$\frac{\alpha + \frac{1}{2} + n}{q[n] + q[n+1]} + \frac{\alpha - \frac{1}{2} + n}{q[n-1] + q[n]} = -2q^2[n] - t. \quad (\text{A.3})$$

The full set of forward and backward difference equations are [23]

$$q[n-1] = -q[n] + \frac{\alpha - \frac{1}{2} + n}{p[n] - 2q[n]^2 - t}, \quad (\text{A.4})$$

$$q[n+1] = -q[n] - \frac{\alpha + \frac{1}{2} + n}{p[n]}, \quad (\text{A.5})$$

$$p[n-1] = -p[n] + 2q[n]^2 + t, \quad (\text{A.6})$$

$$p[n+1] = t - p[n] + 2 \left(q[n] + \frac{\alpha + \frac{1}{2} + n}{p[n]} \right)^2. \quad (\text{A.7})$$

In addition one should note that $H[n+1] = H[n] - q[n+1]$.

Proposition 11. *The solution of the second Painlevé equation as given by (3.4) with parameter $\alpha_1 = 2$ and boundary condition (3.36) is generated from the Hastings–McLeod solution by application of the T_2 Schlesinger transformation applied twice and the Gambier transformation (A.1).*

Proof. Firstly we recall that the parameter for the Hastings–McLeod solution is $\alpha = 0$, $\alpha_1 = 1/2$ whereas we have the case of $\alpha = 3/2$, $\alpha_1 = 2$. Let $\tau = -2^{-1/3}t$. The leading, and defining, asymptotics of the Hastings–McLeod solution at $\alpha = 0$, $\alpha_1 = 1/2$ as $\tau \rightarrow +\infty$ is (for $\xi = 1$, equation (9.47) of [11])

$$q(\tau; \alpha_1 = 1/2) \underset{\tau \rightarrow \infty}{\sim} \text{Ai}(\tau).$$

Using the inverse Gambier transformation (A.1) with $\epsilon = -1$ we have the solution as

$$q(t; \alpha_1 = 0) \underset{t \rightarrow -\infty}{\sim} -2^{-1/3} \frac{\text{Ai}'(-2^{-1/3}t)}{\text{Ai}(-2^{-1/3}t)},$$

and therefore $p(t; \alpha_1 = 0) \sim 0$ and $H(t; \alpha_1 = 0) \sim 0$ in this regime. Now using the Schlesinger transformations (A.5) and (A.7) we deduce

$$\begin{aligned} H(t; \alpha_1 = 2) &= H(t; \alpha_1 = 1) + q(t; \alpha_1 = 1) + \frac{1}{p(t; \alpha_1 = 1)}, \\ &= H(t; \alpha_1 = 0) + \frac{1}{2[q(t; \alpha_1 = 0)]^2 - p(t; \alpha_1 = 0) + t}, \\ &\sim \frac{1}{2[q(t; \alpha_1 = 0)]^2 + t}, \\ &\sim 2^{-1/3} \frac{[\text{Ai}(-2^{-1/3}t)]^2}{[\text{Ai}'(-2^{-1/3}t)]^2 + 2^{-1/3}t [\text{Ai}(-2^{-1/3}t)]^2}, \end{aligned}$$

which is asymptotically equivalent to (3.36). □

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