Solutions to Worksheet 4

Exercise 1:
We know that the solution of the Poisson problem $-\Delta u = f$ with boundary conditions

$$u(0, y) = g_1(y), \quad u(1, y) = g_2(y), \quad u_y(x, 0) = g_3(x), \quad \text{and } u(x, 1) = g_4(x) \tag{0.1}$$

reads $u(x, y) = \frac{10}{\pi} e^{-10\left(2(x-0.5)^2+(y-0.5)^2\right)}$. We can directly derive g_1, g_2 and g_4 with the above definition

$$g_1(y) = u(0, y) = \frac{10}{\pi} e^{-10\left(0.5+(y-0.5)^2\right)} = \frac{10}{\pi} e^{-10(y-0.5)^2-5},$$
$$g_2(y) = u(1, y) = \frac{10}{\pi} e^{-10\left(0.5+(y-0.5)^2\right)} = \frac{10}{\pi} e^{-10(y-0.5)^2-5} = g_1(y),$$
$$g_4(x) = u(x, 1) = \frac{10}{\pi} e^{-10\left(2(x-0.5)^2+0.25\right)} = \frac{10}{\pi} e^{-20(x-0.5)^2-2.5}.$$

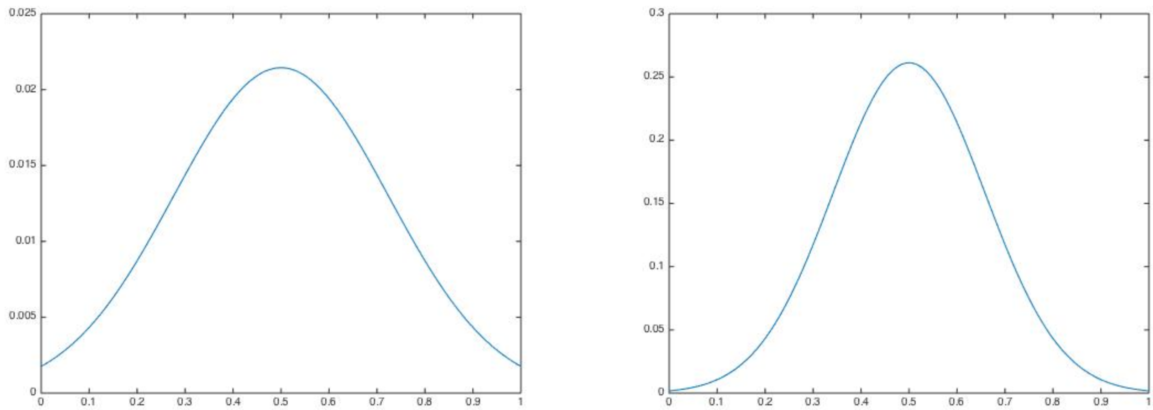


Figure 0.1: The left picture shows g_1, the right g_4

To calculate g_3 and f we need the derivatives of u,

$$u_x(x, y) = u(x, y)(-40(x - 0.5)) = -20(2x - 1)u(x, y),$$
$$u_y(x, y) = u(x, y)(-20(y - 0.5)) = -10(2y - 1)u(x, y),$$
$$u_{xx}(x, y) = u(x, y)(-40(x - 0.5))^2 - 40u(x, y) = 40(40(x - 0.5)^2 - 1)u(x, y),$$
$$u_{yy}(x, y) = u(x, y)(-20(y - 0.5))^2 - 20u(x, y) = 20(20(y - 0.5)^2 - 1)u(x, y).$$

Thus,

$$g_3(x) = u_y(x, 0) = 10u(x, 0) = 10u(x, 1) = \frac{100}{\pi} e^{-20(x-0.5)^2-2.5},$$
$$f(x, y) = -\Delta u(x, y) = -u_{xx}(x, y) - u_{yy}(x, y)$$
$$= -\frac{200}{\pi}(80(x - 0.5)^2 + 20(y - 0.5)^2 - 3)e^{-10\left(2(x-0.5)^2+(y-0.5)^2\right)}$$

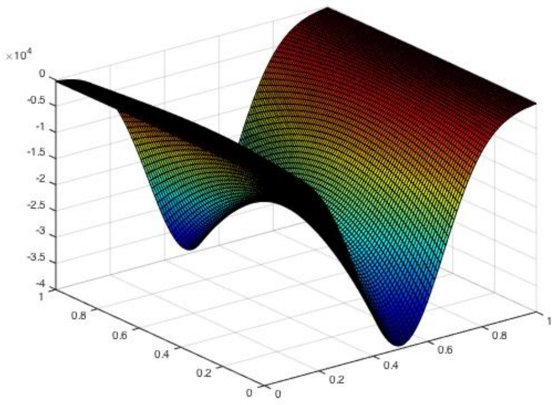
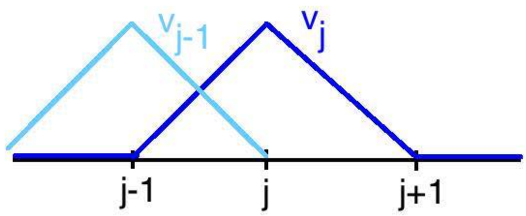


Figure 0.2: f on the domain $\Omega = (0;1) \times (0;1)$

Exercise 2:
We consider a one-dimensional finite-element problem where we discretize an interval $I = [a;b]$ with n nodes $1, ..., j-1, j, j+1, ..., n$. As trial functions we use tent functions:



Our aim is to calculate the corresponding stiffness-matrix

$$m_{i,j} = \int_a^b \partial_x v_i(x) \partial_x v_j(x) \; dx. \tag{0.2}$$

For the derivatives of the tent functions we have

$$\partial_x v_j(x) = \begin{cases} 1 & j-1 < x < j \\ -1 & j < x < j+1 \\ 0 & x < j-1 \vee x > j+1 \end{cases} . \tag{0.3}$$

So, if i and j are not adjacent, i.e. $i \neq j-1, j, j+1$, there is no interval where both derivatives are non-zero, thus

$$m_{i,j} = 0 \text{ for all } i \neq j-1, j, j+1. \tag{0.4}$$

Now we already know that the stiffness-matrix m is a tridiagonal matrix. Furthermore, due to the symmetry of the trial functions, all entries on the diagonal (except for $m_{1,1}$ and $m_{n,n}$) will be equal. The same holds true for both secondary diagonals. So, the only thing that we still have to calculate is $m_{j,j}$ and $m_{j,j+1}$.
We start with $m_{1,1}$, this tent function is cut in half, so

$$m_{1,1} = \int_1^2 (-1)^2 dx = 1.$$

The same holds true for $m_{n,n}$, again, due to the symmetry. Next, for $j \neq 1, n$ we have

$$m_{j,j} = \int_{j-1}^{j} (1)^2 dx + \int_{j}^{j+1} (-1)^2 dx = \int_{j-1}^{j+1} 1 \; dx = 2.$$

At last, let $j \neq n$, then

$$m_{j,j+1} = \int_{j}^{j+1} -1 \; dx = -1.$$

We end up with

$$m = \begin{pmatrix} 1 & -1 & & & 0 \\ -1 & 2 & -1 & & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & & -1 & 2 & -1 \\ 0 & & & -1 & 1 \end{pmatrix}. \tag{0.5}$$

Exercise 3:

Our next exercise is a two-dimensional problem. Our domain is the unit square, $\Omega = [0;1] \times [0;1]$ equipped with the five vertices

$$v_1 = (0,0), \quad v_2 = (1,0), \quad v_3 = (1,1), \quad v_4 = (0,1), \quad v_5 = (0.5, 0.5).$$

These nodes fix a triangulation of Ω into four triangles T_i, $i = 1, ..., 4$.

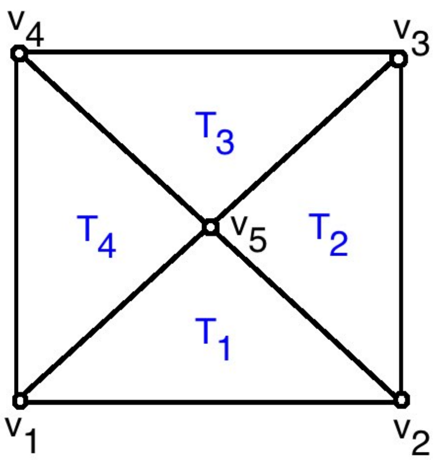


Figure 0.3: Triangulation

We set up linear trial functions of the form $\nu_i(x,y) = a + bx + cy$ for each inner vertex. The i-th trial function satisfies $\nu_i(v_j) = \delta_{i,j}$ and is piecewise linear on Ω. Here, we only need to calculate one trial functions ν_5.
First, we know that $\nu_5(0.5, 0.5) = a + \frac{b}{2} + \frac{c}{2} = 1$ has to hold on all triangles T_i, $i = 1, ..., 4$.
Moreover on T_1 we have $\nu_5(0,0) = a = 0$ and $\nu_5(1,0) = a + b = 0$, so $b = 0$ and $c = 2$,

$$\nu_5(x,y)|_{T_1} = 2y.$$

Repeating this calculation for all triangles, we end up with

$$v_5(x,y) = \begin{cases} 2y & \text{on } T_1 \\ 2-2x & \text{on } T_2 \\ 2-2y & \text{on } T_3 \\ 2x & \text{on } T_4 \end{cases}.$$

Our ansatz for the approximated solution $\tilde{u}$ is a linear combination of all trial functions, here

$$\tilde{u}(x,y) = \alpha\nu_5(x,y).$$

Note that $u(0.5, 0.5) = \alpha$, so α is the value we are looking for.
The weak formulation of the Poisson problem is

$$\int_\Omega \nabla u^T \nabla \nu \ d(x,y) = 4 \int_\Omega \nu \ d(x,y),$$

for all trial functions $\nu \in V$. Since we only have one inner node and thus only one trial function, our approximation $\tilde{u}$ should satisfy

$$\int_\Omega \nabla \tilde{u}^T \nabla \nu_5 \ d(x,y) = 4 \int_\Omega \nu_5 \ d(x,y).$$

Inserting (0.22) we have

$$\alpha \int_\Omega \nabla \nu_5^T \nabla \nu_5 \ d(x,y) = 4 \int_\Omega \nu_5 \ d(x,y),$$

i.e. α is the solution to $A\alpha = F$ with $A = \int_\Omega \nabla \nu_5^T \nabla \nu_5 \ d(x,y)$ and $F = 4\int_\Omega \nu_5 \ d(x,y)$.
For the two integrals, we have

$$\begin{aligned} A &= \int_{T_1} (0\ 2) \begin{pmatrix} 0 \\ 2 \end{pmatrix} \ d(x,y) + \int_{T_2} (-2\ 0) \begin{pmatrix} -2 \\ 0 \end{pmatrix} \ d(x,y) + \int_{T_3} (0\ \ -2) \begin{pmatrix} 0 \\ -2 \end{pmatrix} \ d(x,y) + \int_{T_4} (2\ 0) \begin{pmatrix} 2 \\ 0 \end{pmatrix} \ d(x,y) \\ &= \int_\Omega 4 \ d(x,y) = 4\int_0^1 \int_0^1 1 \ d(x,y) = 4 \end{aligned}$$

$$\begin{aligned} F &= 4\left[\int_{T_1} 2y \ d(x,y) + \int_{T_2} 2-2x \ d(x,y) + \int_{T_3} 2-2y \ d(x,y) + \int_{T_4} 2x \ d(x,y)\right] \\ &= 4\left[\int_0^{\frac{1}{2}} \int_y^{1-y} 2y \ dxdy + \int_{\frac{1}{2}}^1 \int_{1-x}^{x} 2-2x \ dydx + \int_{\frac{1}{2}}^1 \int_{1-y}^{y} 2-2y \ dxdy + \int_0^{\frac{1}{2}} \int_x^{1-x} 2x \ dydx\right] \\ &= 4\left[\int_0^{\frac{1}{2}} 2y(1-2y) \ dy + \int_{\frac{1}{2}}^1 (2-2x)(2x-1) \ dx + \int_{\frac{1}{2}}^1 (2-2y)(2y-1) \ dy + \int_0^{\frac{1}{2}} 2x(1-2x) \ dx\right] \\ &= 8\left[\int_{\frac{1}{2}}^1 (2-2x)(2x-1) \ dx + \int_0^{\frac{1}{2}} 2x(1-2x) \ dx\right] \\ &= 8\left[2\int_{\frac{1}{2}}^1 (2x-1) \ dx + \int_{\frac{1}{2}}^1 2x(1-2x) \ dx + \int_0^{\frac{1}{2}} 2x(1-2x) \ dx\right] \\ &= 8\left[2\int_{\frac{1}{2}}^1 (2x-1) \ dx + \int_0^1 2x(1-2x) \ dx\right] \\ &= 8\left[2\left[x^2 - x\right]_{\frac{1}{2}}^1 + \left[x^2 - \frac{4}{3}x^3\right]_0^1\right] \\ &= 8\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{8}{6} = \frac{4}{3}. \end{aligned}$$

Note that this calculation could be reduced to one integral, since ν_5 is rotational symmetric, i.e.

$$F = 16 \int_{T_1} 2y \; d(x,y) = 16 \int_0^{\frac{1}{2}} 2y(1-2y) \; dy = 16 \left[y^2 - \frac{4}{3} y^3 \right]_0^{\frac{1}{2}} = \frac{4}{3}.$$

All in all, $4\alpha = \frac{4}{3}$, $\alpha = \frac{1}{3}$.

$$\tilde{u}(0.5, 0.5) = \frac{1}{3}$$

The entire approximated solution is

$$\tilde{u}(x,y) = \begin{cases} \frac{2}{3}y & \text{on } T_1 \\ \frac{2}{3} - \frac{2}{3}x & \text{on } T_2 \\ \frac{2}{3} - \frac{2}{3}y & \text{on } T_3 \\ \frac{2}{3}x & \text{on } T_4 \end{cases}.$$

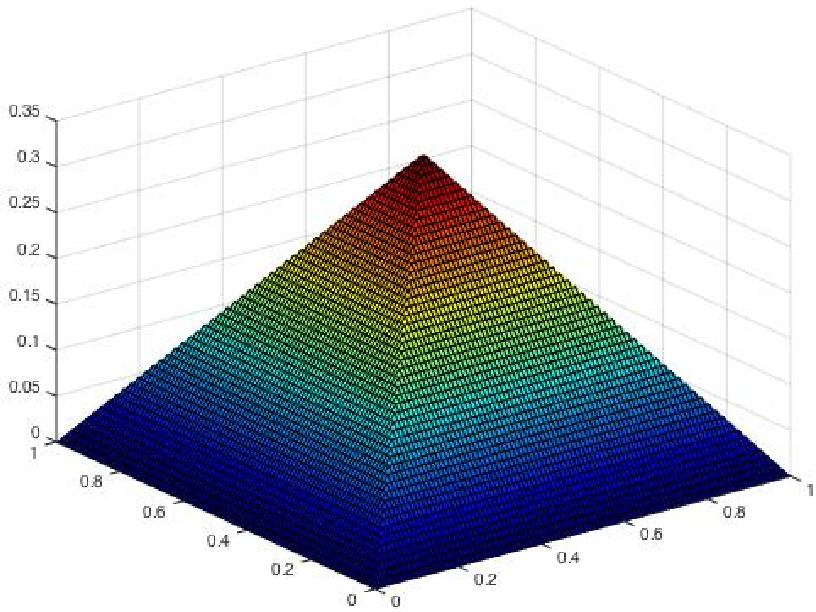


Figure 0.4: Approximated solution