Solutions to Worksheet 9

Exercise 1:
As a predictor for the BDF(2)-method, we use here the parabola q^n determined by the three nodes y_{n-1}, y_n, y_{n+1} and use

$$y_{n+2} = q^n(t_{n+2}) \tag{0.1}$$

as predicted value.

- As showed on worksheet 8 for a uniform grid with stepsize h we can write q^n in the form

  $$q^n(t) = y_{n-1}\frac{(t-t_n)(t-t_{n+1})}{2h^2} - y_n\frac{(t-t_{n-1})(t-t_{n+1})}{h^2} + y_{n+1}\frac{(t-t_{n-1})(t-t_n)}{2h^2}.$$

  Then, inserting t_{n+2} gives

  $$y_{n+2} = y_{n-1}\frac{(t_{n+2}-t_n)(t_{n+2}-t_{n+1})}{2h^2} - y_n\frac{(t_{n+2}-t_{n-1})(t_{n+2}-t_{n+1})}{h^2} + y_{n+1}\frac{(t_{n+2}-t_{n-1})(t_{n+2}-t_n)}{2h^2}$$

  and with the uniform stepsize $t_{n+k} - t_n = kh$ this simplifies to

  $$\begin{aligned} y_{n+2} &= y_{n-1}\frac{2h^2}{2h^2} - y_n\frac{3h^2}{h^2} + y_{n+1}\frac{6h^2}{2h^2} \\ &= y_{n-1} - 3y_n + 3y_{n+1}. \end{aligned}$$

  This is already the equation we are looking for, since we can write it as

  $$y_{n+2} - 3y_{n+1} + 3y_n - y_{n-1} = 0,$$

  the predictor does not take f into account at all. (In particular, this means $b_k = 0$ for $k = 0, 1, 2, 3$.)

- With the coefficients

  $$a_0 = -1,\ a_1 = 3,\ a_2 = -3,\ a_3 = 1$$

  the stability polynomial reads

  $$p(\omega) = -1 + 3\omega - 3\omega^2 + \omega^3 = (\omega - 1)^3.$$

  So, we can directly conclude that all roots are equal to one without further calculations.

- As indicated before, for the approximate solution we first write down the parabola through the initial values y_0, y_1 and y_2. This parabola is uniquely determined and can be written as

  $$q^1(t) = y_0\frac{(t-h)(t-2h)}{2h^2} - y_1\frac{t(t-2h)}{h^2} + y_2\frac{t(t-h)}{2h^2}.$$

  where we used that $t_0 = 0, t_1 = h$ and $t_2 = 2h$. To get the next (predicted) node y_3 we insert $t_3 = 3h$. It is clear that this point will again lie on the parabola q^1.
  Thus, if we calculate the parabola q^2 through the points y_1, y_2, y_3 we will find q^1 again (Remember that there exists only one parabola through three points). Applying this argument repeatedly, we see that all points y_n lie on the parabola and satisfy

  $$y_n = q^1(t_n).$$

So, to take the limit it suffices to take the limit of the parabola. For this purpose we write q^1 in the form $q^1(t) = at^2 + bt + c$,

$$\begin{aligned} q^1(t) &= \frac{1}{2h^2} y_0 (t^2 - 3ht + 2h^2) - \frac{1}{h^2} y_1 (t^2 - 2ht) + \frac{1}{2h^2} y_2 (t^2 - ht) \\ &= t^2 (\frac{y_0}{2h^2} - \frac{y_1}{h^2} + \frac{y_2}{2h^2}) + t(-\frac{3h}{2h^2} y_0 + \frac{2h}{h^2} y_1 - \frac{h}{2h^2} y_2) + y_0 \\ &= \frac{1}{2h^2} t^2 (y_0 - 2y_1 + y_2) + \frac{1}{2h} t(-3y_0 + 4y_1 - y_2) + y_0 \\ &= \frac{1}{2h^2} t^2 (1 - 2e^{h\lambda} + e^{2h\lambda}) + \frac{1}{2h} t(-3 + 4e^{h\lambda} - e^{2h\lambda}) + 1 \\ &= \frac{(e^{h\lambda} - 1)^2}{2h^2} t^2 + \frac{-(e^{h\lambda} - 2)^2 + 1}{2h} t + 1 \end{aligned}$$

Thus, to take the limit we have to calculate the limit of the two fractions first. With L'Hôpital's rule we get

$$\begin{aligned} \lim_{h \to 0} \frac{(e^{h\lambda} - 1)^2}{2h^2} &= \lim_{h \to 0} \frac{2(e^{h\lambda} - 1) e^{h\lambda} \lambda}{4h} \\ &= \lim_{h \to 0} \frac{\lambda^2 e^{2h\lambda} + (e^{h\lambda} - 1) e^{h\lambda} \lambda^2}{2} = \frac{\lambda^2}{2} \end{aligned}$$

and

$$\lim_{h \to 0} \frac{-(e^{h\lambda} - 2)^2 + 1}{2h} = \lim_{h \to 0} \frac{-2(e^{h\lambda} - 2) e^{h\lambda} \lambda}{2} = \lambda.$$

All in all,

$$\lim_{h \to 0} q^1(t) = \frac{\lambda^2}{2} t^2 + \lambda t + 1.$$

To get a better interpretation of this result, we note that the exact solution to $\dot{y}(t) = \lambda y(t)$ with $y(0) = 1$ is given by $y(t) = e^{\lambda t}$. The quadratic form above is just the Taylor expansion up to the second order of this function or, equivalently, the first three terms of the exponential series

$$e^{\lambda t} = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} t^k.$$

Short proof:

We can write any quadratic polynomial as

$$p(t) = p(0) + p'(0)t + \frac{1}{2} p''(0) t^2 \tag{0.2}$$

(Tayler expansion around 0 up to second order). Let p be the polynomial that contains y_0, y_1 and y_2. With the differential quotient, the derivatives are

$$p'(0) = \lim_{h \to 0} \frac{p(h) - p(0)}{h} = \lim_{h \to 0} \frac{y(h) - y(0)}{h} = y'(0)$$

and moreover since p is quadratic

$$p''(0) = \lim_{h \to 0} \frac{p(2h) + p(0) - 2p(h)}{h^2} = \lim_{h \to 0} \frac{y(2h) + y(0) - 2y(h)}{h} = y''(0).$$

Thus, $p(t) = y(0) + y'(0)t + \frac{1}{2}y''(0)t^2$ is the second Taylor polynomial of the exact solution y around 0. One can moreover proof by induction that $p = q_n$ is the polynomial that is used in every step to make the prediction.
From our definition it is clear that $p = q_0$.
For general n one can show that $q_n = q_{n+1}$ with the same argument as above,

$$q_n(t_{n+1}) = y_{n+1} = q_{n+1}(t_{n+1})$$
$$q_n(t_{n+2}) = y_{n+1} = q_{n+1}(t_{n+2})$$
$$q_n(t_{n+2}) = y_{n+1} = q_{n+1}(t_{n+2})$$

and if two parabolas have 3 intersection points, they are equal.

Exercise 3:
We consider the heat equation

$$u_t(x,t) - u_{xx}(x,t) = \sin(x) \tag{0.3}$$

for $x \in [0;\pi]$, $t \in [0;T]$ with initial values

$$u(x,0) = u(0,t) = u(\pi,t) = 0. \tag{0.4}$$

- To get the exact solution of Equation (0.3) & (0.4), we use the Fourier series as ansatz, i.e.

$$u(x,t) = \sum_{k=1}^{\infty} \alpha_k(t)\sin(kx) + \sum_{k=0}^{\infty} \beta_k(t)\cos(kx).$$

With this definition, we find for the partial derivatives

$$u_t(x,t) = \sum_{k=0}^{\infty} \alpha_k'(t)\sin(kx) + \beta_k'(t)\cos(kx),$$
$$u_{xx}(x,t) = -\sum_{k=0}^{\infty} \alpha_k(t)k^2\sin(kx) + \beta_k(t)k^2\cos(kx).$$

Inserting in Equation (0.3) then yields

$$\sum_{k=1}^{\infty}(\alpha_k'(t) + k^2\alpha_k(t))\sin(kx) + \sum_{k=0}^{\infty}(\beta_k'(t) + k^2\beta_k(t))\cos(kx) = \sin(x).$$

Comparing the coefficients (note that the Fourier expansion is unique, since $\sin(kx)$ and $\cos(kx)$ form an ONB on $[0;\pi]$) leads to

$$\alpha_1'(t) + \alpha_1(t) = 1$$

while

$$\alpha_k'(t) + k^2\alpha_k(t) = 0 \quad \text{for } k > 1,$$
$$\beta_k'(t) + k^2\beta_k(t) = 0 \quad \text{for } k \geq 0.$$

From the initial values (0.4) we moreover get

$$u(x,0) = \sum_{k=1}^{\infty} \alpha_k(0)\sin(kx) + \sum_{k=0}^{\infty} \beta_k(0)\cos(kx) \equiv 0,$$

and thus,

$$\alpha_j(0) = 0 \quad \text{and } \beta_k(0) = 0 \quad \text{for all } j \geq 1 \text{ and } k \geq 0.$$

So, the differential equations for the coefficients are solved by

$$\alpha_1(t) = 1 - e^{-t},$$
$$\alpha_k(t) = 0 \quad \text{for } k > 1,$$
$$\beta_k(t) = 0 \quad \text{for } k \geq 0.$$

and the exact solution is given by

$$u(x,t) = (1 - e^{-t})\sin(x).$$

- With its definition, we can just directly show that $u^n(x)$ satisfies

$$\frac{1}{\tau}(u^n(x) - u^{n-1}(x)) = u^n_{xx}(x) + \sin(x), \quad u^n(0) = u^n(\pi) = 0. \tag{0.5}$$

The initial conditions are fulfilled since $\sin(0) = \sin(\pi) = 0$.
Furthermore, since $\frac{d^2}{dx^2}\sin(x) = -\sin(x)$, we have for the derivative

$$u^n_{xx}(x) = -(1 - \frac{1}{(1+\tau)^n})\sin(x) = (-1 + \frac{1}{(1+\tau)^n})\sin(x).$$

So, the right hand side of Equation (0.5) equals

$$u^n_{xx}(x) + \sin(x) = \frac{1}{(1+\tau)^n}\sin(x).$$

Otherwise, on the left hand side, we find

$$\begin{aligned}\frac{1}{\tau}(u^n(x) - u^{n-1}(x)) &= \frac{1}{\tau}\left(1 - \frac{1}{(1+\tau)^n} - 1 + \frac{1}{(1+\tau)^{n-1}}\right)\sin(x)\\ &= \frac{1}{\tau}\left(\frac{1}{(1+\tau)^{n-1}} - \frac{1}{(1+\tau)^n}\right)\sin(x)\\ &= \frac{1}{\tau}\left(\frac{1+\tau}{(1+\tau)^n} - \frac{1}{(1+\tau)^n}\right)\sin(x)\\ &= \frac{1}{\tau}\cdot\frac{\tau}{(1+\tau)^n}\sin(x)\\ &= \frac{1}{(1+\tau)^n}\sin(x).\end{aligned}$$

This proofs the claim.

- To show pointwise convergence, we have to show that

$$\lim_{\tau\to 0} u^N(x) = u(x,T) \quad \text{for all } x \in [0;\pi] \tag{0.6}$$

with $\tau = \frac{T}{N}$. Using the previous results we have to show that

$$\lim_{\tau\to 0}\left(1 - \frac{1}{(1+\tau)^N}\right)\sin(x) = (1 - e^{-T})\sin(x) \quad \text{for all } x \in [0;\pi]\,.$$

This is equivalent to

$$\lim_{\tau\to 0}\left(1 - \frac{1}{(1+\tau)^N}\right) = (1 - e^{-T}).$$

(We can basically skip the dependence on x.). To show that the limit equals the exponential function, we use the equality

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x. \tag{0.7}$$

Thus, we have

$$(1 - e^{-T}) = 1 - \lim_{N \to \infty} \frac{1}{(1 + \frac{T}{N})^N}$$

Furthermore, since $\tau = \frac{T}{N}$, $N \to \infty$ is equivalent to $\tau \to 0$ and

$$\begin{aligned}(1 - e^{-T}) &= 1 - \lim_{N \to \infty} \frac{1}{(1 + \tau)^N} \\ &= 1 - \lim_{\tau \to 0} \frac{1}{(1 + \tau)^N} \\ &= \lim_{\tau \to 0} 1 - \frac{1}{(1 + \tau)^N}.\end{aligned}$$