Numerical Programming 2 (CSE) 2015

Worksheet 9

Exercise 1 (Predictor of the BDF(2) method)

The predictor of the BDF(2) method with uniform time grid $t_n = nh$ is given by $y^0_{n+2} = p_n(t_{n+2})$, where $p_n \in \mathbb{P}_2$ is the interpolation polynomial fulfilling $p_n(t_k) = y_k$ for $k = n-1, n, n+1$. By defining $\widetilde{n} := n - 1$, $y_{\widetilde{n}+3} := y^0_{n+2}$, the predictor value can be seen as the result of a three step method

$$\sum_{k=0}^{3} a_k y_{\widetilde{n}+k} = h \sum_{k=0}^{3} b_k f(t_{\widetilde{n}+k}, y_{\widetilde{n}+k}).$$

- Find the coefficients of this three step method.
- Find the roots of this method's stability polynomial

$$w \mapsto \sum_{k=0}^{3} (a_k - z b_k) w^k.$$

- Consider the application of this method to the test equation $\dot{y} = \lambda y$, $y(0) = 1$, $\lambda \in \mathbb{R}$ with exact initial values $y_n = e^{nh\lambda}$, $n = 0, 1, 2$. Find the limit of the approximate solution computed by the method as $h \to 0$.

  Hint: Solving this problem requires hardly any computation if you find the right approach.

Exercise 2 (Method of lines)

Consider the heat equation with source term

$$u_t(x,t) = u_{xx}(x,t) + f(x), \qquad x \in [0,1],\ t \in [0,1].$$

Discretize the equation in space using finite differences on a grid with $N + 1$ points to obtain a system of ODEs for the values $u_j(t) \approx u(jh, t)$, $j = 0, \ldots, N$, $h := 1/N$.

Perform the time integration of these ODEs by using the built-in MATLAB functions (ode23, ode23s or ode45 for instance) together with the following data

- $f(x) = 10x, \quad u(0,t) = (4x(1-x))^8, \quad$ and $\quad u(0,t) = u(1,t) = 0$
- $f(x) = 50(x-1)^2, \quad u(0,t) = 2(4x(1-x))^8, \quad$ and $\quad u_x(0,t) = u_x(1,t) = 0$

Create a plot showing the relationship between the spatial resolution h and the required number of time steps for different integrators.

Exercise 3 (Rothe Method)

Consider the heat equation

$$u_t(x,t) - u_{xx}(x,t) = \sin(x), \qquad x \in [0,\pi],\ t \in [0,T]$$

with initial and boundary condition given respectively by $u(x,0) = u(0,t) = u(\pi,t) = 0$.

- Using Fourier expansion find the exact solution of this equation.
- By applying the implicit Euler method with step size $\tau = T/N$ we obtain $u^n(x) \approx u(x,n\tau)$ as the solution of the second order (in the space variable) ODE:

$$\begin{cases} \dfrac{u^n(x) - u^{n-1}(x)}{\tau} = u^n_{xx}(x) + \sin(x) \\ u^n(0) = u^n(\pi) = 0 \end{cases} \tag{1}$$

  Using induction prove that the solution of (1) is given by

$$u^n(x) = \left(1 - \frac{1}{(1+\tau)^n}\right)\sin(x).$$

- Show that the numerical solution $u^N(x)$ converges pointwise to the exact solution $u(x,T)$ as $\tau \to 0$.