Solutions to Worksheet 8

Exercise 1:
We are trying to solve the ODE

$$\dot{y}(t) = f(t, y) \tag{0.1}$$

with the BDF(2)-method, i.e. we take a quadratic polynomial as ansatz.

- Let $t_1, t_2, \ldots$ be an arbitrary grid. We start and set up a quadratic polynomial q that satisfies

$$q(t_{n+j}) = y_{n+j} \quad \text{for } j = 0, 1, 2. \tag{0.2}$$

With three conditions a quadratic polynomial is uniquely determined. A standard ansatz is to write q in the form

$$q(t) = y_n \frac{(t - t_{n+1})(t - t_{n+2})}{(t_n - t_{n+1})(t_n - t_{n+1})} + y_{n+1} \frac{(t - t_n)(t - t_{n+2})}{(t_{n+1} - t_n)(t_{n+1} - t_{n+2})} + y_{n+2} \frac{(t - t_n)(t - t_{n+1})}{(t_{n+2} - t_n)(t_{n+2} - t_{n+1})}.$$

It can easily be seen that this polynomial satisfies (0.2). Moreover, for the BDF(2)-method we require that the differential equation (0.1) is fulfilled in the node t_{n+2}, i.e.

$$q'(t_{n+2}) = f(t_{n+2}, y_{n+2}).$$

With our ansatz we can directly calculate the first derivative as

$$q'(t) = y_n \frac{2t - t_{n+1} - t_{n+2}}{(t_n - t_{n+1})(t_n - t_{n+1})} + y_{n+1} \frac{2t - t_n - t_{n+2}}{(t_{n+1} - t_n)(t_{n+1} - t_{n+2})} + y_{n+2} \frac{2t - t_n - t_{n+1}}{(t_{n+2} - t_n)(t_{n+2} - t_{n+1})}.$$

Inserting t_{n+2} yields

$$\begin{aligned} q'(t_{n+2}) &= y_n \frac{t_{n+2} - t_{n+1}}{(t_n - t_{n+1})(t_n - t_{n+1})} + y_{n+1} \frac{t_{n+2} - t_n}{(t_{n+1} - t_n)(t_{n+1} - t_{n+2})} + y_{n+2} \frac{2t_{n+2} - t_n - t_{n+1}}{(t_{n+2} - t_n)(t_{n+2} - t_{n+1})} \\ &\overset{!}{=} f(t_{n+2}, y_{n+2}). \end{aligned}$$

This equation is already of the desired form, but we can simplify it a little bit further by canceling the fractions.

$$\begin{aligned} &(t_{n+2} - t_n)(t_{n+1} - t_n)(t_{n+2} - t_n) f(t_{n+2}, y_{n+2}) = \\ &\qquad (t_{n+2} - t_{n+1})^2 y_n - (t_{n+2} - t_n)^2 y_{n+1} + (t_{n+2} + t_{n+2} - t_n - t_{n+1})(t_{n+1} - t_n) y_{n+2} \end{aligned} \tag{0.3}$$

- For a uniform grid with stepsize h, we know that $t_{n+1} - t_n = t_{n+2} - t_{n+1} = h$ and $t_{n+2} - t_n = 2h$. Inserting this into (0.3) simplifies things further:

$$2h^3 f(t_{n+2}, y_{n+2}) = h^2 y_n - 4h^2 y_{n+1} + 3h^2 y_{n+2}.$$

Dividing by $3h^2$ already gives the desired equality.

$$\frac{2}{3} h f(t_{n+2}, y_{n+2}) = \frac{1}{3} y_n - \frac{4}{3} y_{n+1} + y_{n+2}.$$

- To show the stability result, we first have to read off the coefficients from the equation above. We find

$$a_0 = \frac{1}{3},\ a_1 = -\frac{4}{3},\ a_2 = 1,\ b_0 = 0,\ b_1 = 0,\ b_2 = \frac{2}{3},$$

and thus the stability polynomial

$$p(\omega) = (a_0 - zb_0) + (a_1 - zb_1)\omega + (a_2 - zb_2)\omega^2$$

in this case reads

$$p(\omega) = \frac{1}{3} - \frac{4}{3}\omega + (1 - \frac{2}{3}z)\omega^2.$$

To show A(0)-stability, we have to proof that the roots of this polynomial are smaller than 1. Since $p(\omega)$ is a quadratic polynomial, we can determine its roots with the quadratic formula. (Note that $p(\omega)$ also depends on z, so the roots ω_1 and ω_2 will depend on z, too.) We have,

$$\begin{aligned}\omega_{1/2} &= \frac{\frac{4}{3} \pm \sqrt{\frac{16}{9} - 4 \cdot \frac{1}{3} \cdot (1 - \frac{2}{3}z)}}{2(1 - \frac{2}{3}z)} = \frac{\frac{4}{3} \pm \sqrt{\frac{4}{9} + \frac{8}{9}z}}{2(1 - \frac{2}{3}z)} = \frac{\frac{4}{3} \pm \frac{2}{3}\sqrt{1 + 2z}}{2(1 - \frac{2}{3}z)} \\ &= \frac{2 \pm \sqrt{1 + 2z}}{3 - 2z}.\end{aligned}$$

Next, we assume that $z \in \mathbb{R}$ and $z \leq 0$ and estimate the absolute value of $\omega_1(z)$ first. Since we want to take the square root of a term that includes z, it is easier if we distinguish three cases:

1. Let $0 \geq z > -\frac{1}{2}$. In this case $1 + 2z > 0$ and $\omega_1(z) \in \mathbb{R}$. So,

$$|\omega_1(z)| = \frac{2 + \sqrt{1 + 2z}}{3 - 2z} \leq \frac{2 + \sqrt{1 + 2z}}{3} \leq \frac{2 + 1}{3} = 1,$$

   where we used that a fraction gets larger if the numerator gets larger or the denominator gets smaller.

2. For $z = -\frac{1}{2}$ we can calculate $\omega_1(z)$ directly and find

$$\omega_1(-\frac{1}{2}) = \frac{2}{3 + 1} = \frac{1}{2} < 1.$$

3. In the last case $z < -\frac{1}{2}$ the discriminant $1 + 2z$ is negativ, so the square root yields a complex number,

$$\omega_1(z) = \frac{2 + \sqrt{-(1 + 2z)}\ i}{3 - 2z}.$$

   Now we have to calculate the absolute value using $|\omega_1|^2 = \omega_1 \cdot \bar{\omega}_1 = \mathrm{Re}(\omega_1)^2 + \mathrm{Im}(\omega_1)^2$,

$$\begin{aligned}|\omega_1(z)| &= \frac{2 + \sqrt{-(1 + 2z)}\ i}{3 - 2z} \cdot \frac{2 - \sqrt{-(1 + 2z)}\ i}{3 - 2z} = \frac{2^2 + (\sqrt{-(1 + 2z)})^2}{(3 - 2z)^2} \\ &= \frac{4 - (1 + 2z)}{(3 - 2z)^2} = \frac{3 - 2z}{(3 - 2z)^2} = \frac{1}{3 - 2z} < 1,\end{aligned}$$

   since for $z < 0$, we have $3 - 2z > 3$.

For the second root ω_2 the proof works similar.

1. For $0 \geq z > -\frac{1}{2}$, we have $\omega_2(z) \in \mathbb{R}$ and

$$|\omega_2(z)| = \frac{2 - \sqrt{1 + 2z}}{3 - 2z} \leq \frac{2 - \sqrt{1 + 2z}}{3} < \frac{2 - 0}{3} = \frac{2}{3} < 1.$$

2. For $z = -\frac{1}{2}$ we find

$$\omega_2(-\frac{1}{2}) = \frac{2}{3+1} = \frac{1}{2} < 1.$$

3. In the case $z < -\frac{1}{2}$ we can write

$$\omega_2(z) = \frac{2 - \sqrt{-(1+2z)}\; i}{3-2z} = \bar{\omega}_1(z)$$

   and thus, the absolute value of ω_2 equals the absolute value of ω_1.