Solutions to Worksheet 7

Exercise 1:

- The 2-step implicite Runge-Kutta-scheme is given by

  $$y_{n+1} = y_n + h(b_1 k_1 + b_2 k_2) \quad \text{where } k_i = f(t_n + c_i h, y_n + h(a_{i,1} k_1 + a_{i,2} k_2))$$

  or all together

  $$y_{n+1} = y_n + h(b_1 f(t_n + c_1 h, y_n + h(a_{1,1} k_1 + a_{1,2} k_2)) + b_2 f(t_n + c_2 h, y_n + h(a_{2,1} k_1 + a_{2,2} k_2))). \quad (0.1)$$

  The associated Butcher-Tableau

  $$\begin{array}{c|cc} c_1 & a_{1,1} & a_{1,2} \\ c_2 & a_{2,1} & a_{2,2} \\ \hline & b_1 & b_2 \end{array}$$

  contains all the coefficients. If we compare the θ-methode

  $$y_{n+1} = y_n + h(\theta f(y_n) + (1 - \theta) f(y_{n+1}))$$

  to Equation (0.1) we immediately find $b_1 = \theta$ and $b_2 = 1 - \theta$. Moreover, $a_{1,1} = a_{1,2} = 0$. To deduce $a_{2,1}$ and $a_{2,2}$ we re-insert the definition of y_{n+1} and find

  $$y_{n+1} = y_n + h(\theta f(y_n) + (1 - \theta) f(y_n + h(\theta f(y_n) + (1 - \theta) f(y_{n+1})))).$$

  Thus, $a_{2,1} = \theta$ and $a_{2,2} = 1 - \theta$ and the Butcher tableau reads

  $$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & \theta & 1-\theta \\ \hline & \theta & 1-\theta \end{array}$$

- One deduces the stability function by applying the scheme to the linear equation $y' = \lambda y$ and the initial value $y_0 = 1$. Then, the stability function R is defined as $y_{n+1} = R(h\lambda) y_n$. If we apply the θ- method once, we find

  $$y_1 = y_0 + h(\theta \lambda y_0 + (1 - \theta) \lambda y_1).$$

  We can rewrite this equation as

  $$\begin{aligned} y_1 &= 1 + h\theta\lambda + h(1 - \theta)\lambda y_1, \\ (1 + h(\theta - 1)\lambda) y_1 &= 1 + h\theta\lambda, \\ y_1 &= \frac{1 + h\theta\lambda}{1 + h(\theta - 1)\lambda}. \end{aligned}$$

  So, with $z = h\lambda$, we directly see that

  $$R(z) = \frac{1 + \theta z}{1 + (\theta - 1) z}.$$

- A method is called "A-stable" if the stability function fulfills

$$|R(z)| \leq 1 \quad \text{for all } z \text{ with } \mathrm{Re}(z) \leq 0. \tag{0.2}$$

For the absolute value of a complex number we write $|z|^2 = z\bar{z}$ and then,

$$|R(z)|^2 = \frac{|1+\theta z|^2}{|1+(\theta-1)z|^2}$$

We follow the hint and use the notation $z = a + ib$,

$$|R(a+ib)|^2 = \frac{|1+\theta a + i\theta b|^2}{|1+(\theta-1)a + i(\theta-1)b|^2}.$$

Obviously $|R(z)|^2 \leq 1$ (and then also $|R(z)| \leq 1$) if

$$|1+\theta a + i\theta b|^2 \leq |1+(\theta-1)a + i(\theta-1)b|^2,$$

so we have to find all θ such that this equation holds true for all $a, b \in \mathbb{R}$ with $a \leq 0$.
On the left hand side we have

$$|1+\theta a + i\theta b|^2 = (1+\theta a)^2 + (\theta b)^2 = 1 + 2\theta a + \theta^2(a^2+b^2),$$

where we used that $|z|^2 = \mathrm{Re}(z)^2 + \mathrm{Im}(z)^2$. On the right hand side, we see

$$|1+(\theta-1)a + i(\theta-1)b|^2 = (1+(\theta-1)a)^2 + ((\theta-1)b)^2 = 1 + 2(\theta-1)a + (\theta-1)^2(a^2+b^2).$$

Thus, we can simplify the inequality to:

$$\begin{aligned} 1 + 2\theta a + \theta^2(a^2+b^2) &\leq 1 + 2(\theta-1)a + (\theta^2 - 2\theta + 1)(a^2+b^2) \\ 0 &\leq -2a + (-2\theta+1)(a^2+b^2) \\ 2a &\leq (1-2\theta)a^2 + b^2 \end{aligned}$$

Now, we want to distinguish the three different cases.

1. We start with the case $\theta = \frac{1}{2}$. Then,

   $$2a \leq 0$$

   holds true for all z with $a < 0$.
2. For $\theta < \frac{1}{2}$, $1 - 2\theta > 0$ and so

   $$2a \leq (1-2\theta)(a^2+b^2)$$

   is satisfied for $a < 0$.
3. In the case $\theta > \frac{1}{2}$, the above inequation is not fulfilled for all $z = a + ib$ with $a < 0$. Take for example $a = -\frac{1}{2}$, then

   $$\begin{aligned} -1 &\leq (1-2\theta)(0,25+b^2), \\ \frac{1}{2\theta-1} &\leq 0,25 + b^2 > b^2. \end{aligned}$$

   what is obviously not true for all $b \in \mathbb{R}$.

So, the θ- method is A-stable for $\theta \in [0; \frac{1}{2}]$.

- A method is called "L-stable" if the stability function fulfills

$$\lim_{z\to-\infty} R(z) = 0. \tag{0.3}$$

Here, we have

$$\lim_{z\to-\infty} R(z) = \lim_{z\to-\infty} \frac{\frac{1}{z}+\theta}{\frac{1}{z}+(\theta-1)} = \frac{\theta}{(\theta-1)}.$$

The θ- method is only L-stable if $\theta = 0$.

Exercise 2:
We consider a dissipative ODE, i.e.

$$\dot{y} = f(y) \tag{0.4}$$

where f satisfies

$$\mathrm{Re}\langle f(x) - f(y), x - y\rangle \leq 0 \quad \text{for all } x, y. \tag{0.5}$$

- The first task is to show that the exact flow of such a dissipative ODE is non-expansive, that means the flow Ψ^t satisfies

$$\|\Psi^t(x) - \Psi^t(y)\| \leq \|x - y\| \quad \text{for all } x, y \text{ and for all } t \geq 0. \tag{0.6}$$

To start, let us note that since $\Psi^0(x) = x$ for all x, the estimate is obviously fulfilled for $t = 0$,

$$\|\Psi^0(x) - \Psi^0(y)\| \leq \|x - y\| \quad \text{for all } x, y.$$

To show that the inequality stays fulfilled for all times $t \geq 0$, it is sufficient to show that the norm is non-increasing, i.e.

$$\partial_t \|\Psi^t(x) - \Psi^t(y)\| \leq 0.$$

To make things a little bit clearer, we begin with the derivative of a norm in general. Let v be an arbitrary differentiable, complex-valued function, then

$$\partial_t \|v(t)\| = \partial_t(\langle v(t), v(t)\rangle)^{\frac{1}{2}} = \frac{1}{2\|v(t)\|}\left(\langle \partial_t v(t), v(t)\rangle + \langle v(t), \partial_t v(t)\rangle\right).$$

Furthermore, a scalar product on $\mathbb{C}$ satisfies $\langle x, y\rangle = \overline{\langle y, x\rangle}$. So,

$$\partial_t \|v(t)\| = \frac{1}{2\|v(t)\|}\left(\langle \partial_t v(t), v(t)\rangle + \overline{\langle \partial_t v(t), v(t)\rangle}\right) = \frac{\mathrm{Re}\langle \partial_t v(t), v(t)\rangle}{\|v(t)\|}.$$

If we apply this to the norm of the flow, we can use that Equation (0.4) is dissipative. We have,

$$\partial_t \|\Psi^t(x) - \Psi^t(y)\| = \frac{\mathrm{Re}\langle \partial_t \Psi^t(x) - \partial_t \Psi^t(y), \Psi^t(x) - \Psi^t(y)\rangle}{\|\Psi^t(x) - \Psi^t(y)\|}.$$

The partial derivative of $\Psi^t(x)$ can be expressed via f, since Ψ^t is the flow of the system (0.4). With $\partial_t \Psi^t(x) = f(\Psi^t(x))$ we can conclude

$$\partial_t \|\Psi^t(x) - \Psi^t(y)\| = \frac{\mathrm{Re}\langle f(\Psi^t(x)) - f(\Psi^t(y)), \Psi^t(x) - \Psi^t(y)\rangle}{\|\Psi^t(x) - \Psi^t(y)\|} \leq 0$$

since f satisfies (0.5) and $\|\Psi^t(x) - \Psi^t(y)\| \geq 0$.
Note that to be precise we have to assume $\Psi^t(x) - \Psi^t(y) \neq 0$. But $\Psi^t(x) = \Psi^t(y)$ if and only if $x = y$ and in this case the estimate is trivially fulfilled.

- Next we want to extend this property to approximated solutions, computed here via the implicit Euler method. As a reminder, for this numerical scheme we use the iterative calculation

$$x_{k+1} = x_k + hf(x_{k+1}), \quad k = 0, 1, \dots . \tag{0.7}$$

to approximate the solution of (0.4).
Our goal is to show that f satisfies (0.5), then the discrete flow $\tilde{\Psi}^h(x) = x + f(\tilde{\Psi}^h(x))$ satisfies

$$\|\tilde{\Psi}^h(x) - \tilde{\Psi}^h(y)\| \leq \|x - y\| \quad \text{for all } x, y \text{ and for all } h \geq 0. \tag{0.8}$$

To get a good estimation for the norm, we start with the definition via the scalar product. Let x, y be arbitrary, then

$$\begin{aligned}\|\tilde{\Psi}^h(x) - \tilde{\Psi}^h(y)\|^2 &= \langle \tilde{\Psi}^h(x) - \tilde{\Psi}^h(y), \tilde{\Psi}^h(x) - \tilde{\Psi}^h(y)\rangle \\ &= \langle x - y + f(\tilde{\Psi}^h(x)) - f(\tilde{\Psi}^h(y)), \tilde{\Psi}^h(x) - \tilde{\Psi}^h(y)\rangle \\ &= \mathrm{Re}\langle x - y, \tilde{\Psi}^h(x) - \tilde{\Psi}^h(y)\rangle + \mathrm{Re}\langle f(\tilde{\Psi}^h(x)) - f(\tilde{\Psi}^h(y)), \tilde{\Psi}^h(x) - \tilde{\Psi}^h(y)\rangle \\ &\leq \mathrm{Re}\langle x - y, \tilde{\Psi}^h(x) - \tilde{\Psi}^h(y)\rangle\end{aligned}$$

since the ODE is dissipative. (The scalar product in the second line has to be real, since it is equal to the real norm. But the two scalar products in the next line could be complex valued, while their imaginary parts have to cancel out. So we can write the sum as the sum of the two real parts.) To get rid of the real part again, we use that

$$\mathrm{Re}\ z \leq |z| = \sqrt{\mathrm{Re}\ z^2 + \mathrm{Im}\ z^2}$$

So, we can bound the norm $\|\tilde{\Psi}^h(x) - \tilde{\Psi}^h(y)\|^2$ with the absolute value of $\langle x - y, \tilde{\Psi}^h(x) - \tilde{\Psi}^h(y)\rangle$ and in order to write this scalar product again as norms, we use the Cauchy-Schwarz-inequality, $|\langle x, y\rangle| \leq \|x\|\|y\|$,

$$\|\tilde{\Psi}^h(x) - \tilde{\Psi}^h(y)\|^2 \leq \|x - y\|\|\tilde{\Psi}^h(x) - \tilde{\Psi}^h(y)\|.$$

Dividing by $\|\tilde{\Psi}^h(x) - \tilde{\Psi}^h(y)\|$ proofs the result.

- As last step, we can show that every B-stable method is also A-stable.
In particular, we have to show that if the discrete flow $\tilde{\Psi}^h$ of any dissipative ODE $\dot{y}(t) = \lambda y(t)$ with $\mathrm{Re}\ \lambda < 0$ satisfies

$$\|\tilde{\Psi}^h(x) - \tilde{\Psi}^h(y)\| \leq \|x - y\| \quad \text{for all } x, y \text{ and for all } t \geq 0, \tag{0.9}$$

then the corresponding stability polynomial $R(z)$ satisfies $|R(z)| \leq 1$.
One can show that the ODE $\dot{y}(t) = \lambda y(t)$ is dissipative if and only if $\mathrm{Re}\ \lambda < 0$ by taking the definition (0.5),

$$\mathrm{Re}\langle f(x) - f(y), x - y\rangle = \mathrm{Re}\langle \lambda(x - y), x - y\rangle = \mathrm{Re}(\bar{\lambda}\|x - y\|) = \mathrm{Re}(\lambda)\|x - y\|.$$

Since $\|x - y\| \geq 0$ for all x, y the expression is negativ (or zero) if and only if $\mathrm{Re}\ \lambda \leq 0$.
The stability function is defined as $y_{n+1} = R(h\lambda)y_n$, thus the discrete flow in this case can be written as $\tilde{\Psi}^h(y) = R(h\lambda)y$. Now, we only have to insert this definition into assumption (0.9) and we immediately see that the claim holds true,

$$\begin{aligned}\|\tilde{\Psi}^h(x) - \tilde{\Psi}^h(y)\| &\leq \|x - y\| \\ \|R(h\lambda)(x - y)\| &\leq \|x - y\| \\ |R(h\lambda)|\|x - y\| &\leq \|x - y\| \\ |R(h\lambda)| &\leq 1\end{aligned}$$