Solutions to Worksheet 5

Exercise 1:
To verify that λ_j and v_j are eigenvalues resp. eigenvectors of M we have to show that

$$Mv_j = \lambda_j v_j \tag{0.1}$$

for all $j = 1, ..., d$ (just by definition of eigenvalues and eigenvectors). The two vectors on both sides are equal if each of their entries coincide, i.e.

$$(Mv_j)_k = \lambda_j v_{j,k}$$

for $k = 1, ..., d$. On the one hand we have

$$\lambda_j v_{j,k} = \sqrt{\frac{2}{d+1}}(\alpha + 2\beta\cos(\frac{\pi j}{d+1}))\sin(\frac{\pi jk}{d+1}).$$

Note that λ_j is just a constant. On the other hand, due to the special structure of M, we only have to take the entries $v_{j,k-1}, v_{j,k}$ and $v_{j,k+1}$ into account,

$$\begin{aligned}(Mv_j)_k &= \beta v_{j,k-1} + \alpha v_{j,k} + \beta v_{j,k+1}\\ &= \sqrt{\frac{2}{d+1}}\left(\beta\sin(\frac{\pi j(k-1)}{d+1}) + \alpha\sin(\frac{\pi jk}{d+1}) + \beta\sin(\frac{\pi j(k+1)}{d+1})\right)\\ &= \sqrt{\frac{2}{d+1}}\left(\alpha\sin(\frac{\pi jk}{d+1}) + \beta(\sin(\frac{\pi j(k-1)}{d+1}) + \sin(\frac{\pi j(k+1)}{d+1}))\right).\end{aligned}$$

This equality holds also true for $j = 1$ and $j = d$ if we define $v_0 = v_{d+1} = 0$. For the sum of two sines we can use the addition theorem $\sin(x) + \sin(y) = 2\sin(\frac{1}{2}(x+y))\cos(\frac{1}{2}(x-y))$:

$$\sin(\frac{\pi j(k-1)}{d+1}) + \sin(\frac{\pi j(k+1)}{d+1}) = 2\sin(\frac{\pi jk}{d+1})\cos(-\frac{\pi j}{d+1})$$

With $\cos(-x) = \cos(x)$, we find

$$\begin{aligned}(Mv_j)_k &= \sqrt{\frac{2}{d+1}}\left(\alpha\sin(\frac{\pi jk}{d+1}) + 2\beta\sin(\frac{\pi jk}{d+1})\cos(\frac{\pi j}{d+1})\right)\\ &= \sqrt{\frac{2}{d+1}}\left(\alpha + 2\beta\cos(\frac{\pi j}{d+1})\right)\sin(\frac{\pi jk}{d+1}).\end{aligned}$$

This proves the first claim. To decompose Equation (6) we first mention that since M is in particular symmetric and real, the eigenvectors $\{v_j\}_{j=1,..,d}$ form an orthonormal set and M can be decomposed into an orthogonal matrix $U \in \mathbb{R}^{d\times d}$ and a diagonal matrix $D \in \mathbb{R}^{d\times d}$ such that

$$M = UDU^T \tag{0.2}$$

with

$$D = \operatorname{diag}(\lambda_1, ..., \lambda_d) \text{ and } U = \begin{pmatrix} | & & | \\ v_1 & \dots & v_d \\ | & & | \end{pmatrix}. \tag{0.3}$$

Since S has tridiagonal Toeplitz structure and $T = \mathrm{Id}$ we can rewrite Equation (6) as

$$x_{l-1} + U_S D_S U_S^T x_l + x_{l+1} = h^2 b_l,$$
$$U_S^T x_{l-1} + D_S U_S^T x_l + U_S^T x_{l+1} = h^2 U_S^T b_l,$$

where we used that $U_S^T U_S = \mathrm{Id}$ since U_S is orthogonal. Consequently, we set $\widetilde{x}_l = U_S^T x_l$ and obtain the decoupled equation

$$\widetilde{x}_{l-1} + D_S \widetilde{x}_l + \widetilde{x}_{l+1} = h^2 U_S^T b_l. \tag{0.4}$$

Since we have for each entry $\widetilde{x}_{l,j}$ of $\widetilde{x}_l$

$$\widetilde{x}_{l-1,j} + \lambda_j \widetilde{x}_{l,j} + \widetilde{x}_{l+1,j} = h^2 (U_S^T b_l)_j. \tag{0.5}$$

there is no correlation between $\widetilde{x}_{l,1}$ and $\widetilde{x}_{l,2}$ etc. anymore. We can also give a more detailed expression for $\widetilde{x}_l$ with the previous formulas for the eigenvalues and eigenvectors. First, we have $\lambda_j = -4 + \cos(\frac{\pi j}{d+1})$. Furthermore,

$$(U_S^T b_l)_j = \sqrt{\frac{2}{d+1}} \sum_{k=1}^{d} \sin(\frac{\pi j k}{d+1}) b_{l,k}$$

and with the notation $\widetilde{x}_l = [\widetilde{u}_{1,l}, \ldots, \widetilde{u}_{d,l}]^T$

$$\widetilde{u}_{j,l} = \sqrt{\frac{2}{d+1}} \sum_{k=1}^{d} \sin(\frac{\pi j k}{d+1}) u_{k,l}.$$