Numerical Programming 2 (CSE) 2015

Worksheet 5

Exercise 1 (Sparse linear systems)

A TST matrix is a Toeplitz, symmetric, tridiagonal matrix[1]. Let M be the following $d \times d$ TST matrix

$$M = \begin{pmatrix} \alpha & \beta & 0 & 0 & ... \\ \beta & \alpha & \beta & 0 & ... \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & ... & \beta & \alpha & \beta \\ 0 & ... & 0 & \beta & \alpha \end{pmatrix} \tag{1}$$

verify that its eigenvalues and eigenvectors are respectively given by

$$\lambda_j = \alpha + 2\beta \cos\left(\frac{\pi j}{d+1}\right) \qquad v_{j,l} = \sqrt{\frac{2}{d+1}} \sin\left(\frac{\pi j l}{d+1}\right) \quad \text{for } j, l = 1, .., d. \tag{2}$$

Consider the five-point stencil for discretizing the 2D Laplacian operator on a square, uniform grid. The structure of the resulting matrix defining the linear system, when using natural ordering, is given by

$$M = \frac{1}{h^2} \begin{pmatrix} S & T & O & ... & O \\ T & S & T & \ddots & \vdots \\ O & \ddots & \ddots & \ddots & O \\ \vdots & \ddots & T & S & T \\ O & ... & O & T & S \end{pmatrix}, \tag{3}$$

where is the all zeros matrix and S and T are both TST matrices:

$$S = \begin{pmatrix} -4 & 1 & 0 & ... & 0 \\ 1 & -4 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & -4 & 1 \\ 0 & ... & 0 & 1 & -4 \end{pmatrix} \quad \text{and} \quad T = Id \tag{4}$$

The linear system to be solved then is given by

$$M \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \tag{5}$$

[1] A matrix is said to be Toeplitz if it is constant along its diagonals.

with $x_l = [u_{1,l}, u_{2,l}, ..., u_{n,l}]^T$ and $b_l = [f_{1,l}, f_{2,l}, ..., f_{n,l}]^T$. Equivalently the system in (5) can be rewritten as

$$Tx_{l-1} + Sx_l + Tx_{l+1} = h^2 b_l \tag{6}$$

saying that the vectors-variables x_l are all coupled to each other.

Use the information given in (2) to rewrite (6) in an equivalent form in which the variables are decoupled, i.e. introduce new variables $\widetilde{x}_l$ such that, e.g., $\widetilde{x}_{l,1}$ and $\widetilde{x}_{l,2}$ are uncoupled.

Exercise 2 (Jacobi iteration)

Discretize the 1D Poisson equation

$$\Delta u(x) = f(x) \quad \text{on} \quad (0,1)$$

with $f(x) = \cos(\pi x)$ and Dirichlet boundary conditions $u(0) = u(1) = 0$ by using three-point Finite Differences on an uniform grid with $h = 1/N$.

For $N = 2^k$, $k = 3, \ldots, 8$, solve the resulting system of linear equations using Jacobi iteration. Note that you will only get approximate solutions $\tilde{u}_h \approx u_h$ of the discrete problem $\Delta_h u_h = f$. Determine experimentally how many iterations you need to obtain an approximation fulfilling

$$\|\Delta_h \tilde{u}_h - f\|_\infty \leq 10^{-3}$$

and plot the relationship between N and the required number of iterations.