Numerical Programming 2 (CSE) 2015

Worksheet 4

Exercise 1

(Finite differences Neumann bc) Consider the following equation on the unit square $[0,1] \times [0,1]$

$$\begin{cases} -\triangle u = f \ \text{ on } \ (0,1) \times (0,1) \\ u(0,y) = g_1(y) \ \text{ for } \ y \in [0,1] \\ u(1,y) = g_2(y) \ \text{ for } \ y \in [0,1] \\ \dfrac{\partial u}{\partial y}(x,0) = g_3(x) \ \text{ for } \ x \in [0,1] \\ u(x,1) = g_4(x) \ \text{ for } \ x \in [0,1] \end{cases} \tag{1}$$

- Determine the functions f and g_i such that the exact solution is

$$u(x,y) = \frac{10}{\pi} e^{-10\left[2(x-0.5)^2+(y-0.5)^2\right]}.$$

- Discretize the domain with a uniform meshgrid with stepsize $h = 1/(N+1)$ and use the five points stencil discretization of the Laplacian. Approximate the Neumann boundary condition at $y = 0$ by

$$\frac{\partial u}{\partial y}(x_i,0) \approx \frac{u(x_i,h) - u(x_i,0)}{h} \quad \text{for} \quad i = 1,2,...,N. \tag{2}$$

  Solve the problem (1) with $N = 20, 40, 80, 160$ and calculate the error for each N. Plot the error as a function of N in a log-log plot. What is the order of convergence?

- By introducing the so called “ghost points” replace eq. (2) with the central difference approximation

$$\frac{\partial u}{\partial y}(x_i,0) \approx \frac{u(x_i,h) - u(x_i,-h)}{2h} \quad \text{for} \quad i = 1,2,...,N. \tag{3}$$

  and repeat the experiment computing the error for $N = 20, 40, 80, 160$. Did you notice any improvements in the order of convergence?

Exercise 2

In one dimension the geometric building blocks of the finite element method are the intervals. Let the trial function $v_j(x)$ be the "tent" function defined by

$$\begin{cases} v_j(x) = 1 - j + x & \text{for} \quad j-1 \leq x \leq j \\ v_j(x) = 1 + j - x & \text{for} \quad j \leq x \leq j+1 \\ v_j(x) = 0 & \text{elsewhere} \end{cases}$$

Compute the one dimensional analog of the stiffness matrix

$$m_{ij} = \int \nabla v_i(x) \cdot \nabla v_j(x) dx$$

Exercise 3

Consider the problem $\triangle u = -4$ in the unit square with $u(0, y) = u(1, y) = u(x, 0) = u(x, 1) = 0$. Partition the square into 4 triangles by drawing its diagonals. Use the finite element method with piece-wise linear trial functions $v_i(x, y) = a + bx + cy$ to find the approximate value $u(0.5, 0.5)$ at the center of the domain.