Numerical Programming 2 (CSE) 2015

Worksheet 3
Updated

Exercise 1

Let f be a smooth function and $h \in \mathbb{R}$, $0 < h \ll 1$. By using Taylor expansion on the terms $f(x+h)$ and $f(x-h)$ deduce the following $O(h^2)$ accurate approximation for the second derivative of f:

$$\frac{d^2 f(x)}{dx^2} \approx \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} \tag{1}$$

- Use the values of $f(x,y)$, $f(x+h,y)$, $f(x,y+h)$, $f(x-h,y)$ and $f(x,y-h)$ to approximate the value of Laplacian operator $\triangle f(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ at the point (x,y).
- Deduce a $O(h^4)$ accurate approximation of the Laplacian operator at the point (x,y) using the values $f(x,y)$, $f(x \pm h, y)$, $f(x, y \pm h)$, $f(x \pm 2h, y)$ and $f(x, y \pm 2h)$.
- Verify that the function $u(x,y) = e^{\pi x} \sin(\pi y) + 0.5(xy)^2$ is the solution of the following boundary value problem defined on the square $[0,1] \times [0,1]$:

$$\begin{cases} \triangle u(x,y) = x^2 + y^2 \\ u(0,y) = \sin(\pi y), \qquad u(x,0) = 0 \\ u(1,y) = e^{\pi} \sin(\pi y) + \dfrac{y^2}{2}, \qquad u(x,1) = \dfrac{x^2}{2}. \end{cases}$$

- On a uniform grid with stepsize h, use the five point stencil formula to compute the numerical solution of the above boundary value problem and plot the error between the numerical solution and the exact solution for different values of h. Verify that the error scales like $O(h^2)$
- Do the same using the nine stencil formula and check that the error scales like $O(h^4)$[1].

[1] Even though it is a bit like cheating, you can use the exact solution to get the extra information about u outside Ω that is needed when inplementing the nine stencil. Can you think to a better way?

Exercise 2

Let $\triangle_h$ be the five stencil approximation of the Laplacian defined on a square domain Ω (take again $[0,1]\times[0,1]$) discretized with a uniform grid Ω_h with stepsize h. Let $\partial\Omega$ and $\partial\Omega_h$ be the boundary of Ω and Ω_h respectively.

- (Discrete Maximum Principle) Prove that, given a function satisfying $\triangle_h v_h \geq 0$ on Ω_h, we have that

$$\max_{(x_i,y_j)\in\Omega_h} v_h(x_i,y_j) \leq \max_{(x_i,y_j)\in\partial\Omega_h} v_h(x_i,y_j),$$

  with equality holding if and only if v is a constant function. (Hint: Can the arithmetic mean of 4 numbers be bigger -smaller- than the 4 numbers?)

- Similarly, assuming that $\triangle_h v_h \leq 0$, show that

$$\min_{(x_i,y_j)\in\Omega_h} v_h(x_i,y_j) \geq \min_{(x_i,y_j)\in\partial\Omega_h} v_h(x_i,y_j),$$

- Use the maximum (minimum) principle to show the uniqueness of the solution of the discrete Laplace boundary value problem

$$\triangle_h v_h = f \text{ on } \Omega_h, \qquad v_h = g \text{ on } \partial\Omega_h \tag{2}$$

  (Hint: Suppose there are two solutions and consider their difference; which equation does the difference satisfy?)

- Prove that the solution of eq. (2) satisfies

$$\max_{\Omega_h}|v_h| \leq \frac{1}{8}\max_{\Omega_h}|f| + \max_{\partial\Omega_h}|g| \tag{3}$$

  (Hint: let $M_f = \max_{\Omega_h}|f|$ and start by computing $\triangle_h(v_h + M_f\phi)$ with $\phi(x,y) = \frac{1}{4}\left[\left(x-\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2\right]$)

- Let v be the solution of

$$\triangle v = f \text{ on } \Omega, \qquad v = g \text{ on } \partial\Omega. \tag{4}$$

  Use eq. (3) to show that

$$\max_{\Omega_h}|v_h - v| \leq \frac{1}{8}\max_{\Omega_h}|\triangle v - \triangle_h v|$$

- Using the consistency error estimates obtained in Exercise 1 (namely $|\triangle v - \triangle_h v| \leq Ch^2$) conclude that $\lim_{h\to 0}|v_h - v| = 0$ showing the convergence of the five stencil scheme.